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HOMOTOPY LIE ALGEBRA OF CLASSIFYING SPACES FOR HYPERBOLIC COFORMAL 2-CONES

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Abstract

In this paper, we show that the rational homotopy Lie algebra of classifying spaces for certain types of hyperbolic coformal 2-cones is not nilpotent.

1. Introduction

A simply connected space X is called an n-cone if it is built up by a sequence of cofibrations

$$Y_k \xrightarrow{f} X_{k-1} \xrightarrow{j_k} X_k$$

with $X_0 = *$ and $X_n \simeq X$. One can further assume that $Y_k \simeq \Sigma^{k-1} W_k$ is a $(k-1)$-fold suspension of a connected space W_k [3]. In particular a 2-cone X is the cofibre of a map between two suspensions

$$\Sigma A \xrightarrow{f} \Sigma B \to X. \tag{1}$$

Spaces under consideration are assumed to be 1-connected and of finite type, that is, $H^i(X;\mathbb{Q})$ is a finite-dimensional $\mathbb{Q}$-vector space. To every space X corresponds a free chain Lie algebra of the form $(\mathbb{L}(V),\delta)$ [2], called a Quillen model of X. It is an algebraic model of the rational homotopy type of X. In particular, one has an isomorphism of Lie algebras $H_*(\mathbb{L}(V),\delta) \cong \pi_*(\Omega X)\otimes\mathbb{Q}$. The model is called minimal if $\delta V \subset \mathbb{L}^{\geqslant 2}(V)$. A space X is called coformal if there is a map of differential Lie algebras $(\mathbb{L}(V),\delta) \to (\pi_*(\Omega X)\otimes\mathbb{Q}, 0)$ that induces an isomorphism in homology. Any continuous map $f : X \to Y$ has a Lie representative $\tilde{f} : (\mathbb{L}(W),\delta') \to (\mathbb{L}(V),\delta)$ between respective models of X and Y.

If X is a 2-cone as defined by (1) and $\tilde{f} : \mathbb{L}(W) \to \mathbb{L}(V)$ is a model of f, then a Quillen model of the cofibre X of f is obtained as the push out of the following diagram:

$$\begin{array}{ccc}
(\mathbb{L}(W),0) & \xrightarrow{\tilde{f}} & (\mathbb{L}(V),0) \\
\downarrow \imath & & \downarrow \bar{\imath} \\
(\mathbb{L}(W\oplus sW),d) & \xrightarrow{\bar{f}} & (\mathbb{L}(V\oplus sW),\delta)
\end{array}$$

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where $(\mathbb{L}(W \oplus sW), d)$ is acyclic. Moreover the differential on $\mathbb{L}(V \oplus sW)$ verifies $\delta sW \subset \mathbb{L}(V)$. Hence a 2-cone X has a Quillen model of the form $(\mathbb{L}(V_1 \oplus V_2), \delta)$ such that $\delta V_1 = 0$ and $\delta V_2 \subset \mathbb{L}(V_1)$.

A Sullivan model of a space X is a cochain algebra $(\wedge Z, d)$ that algebraically models the rational homotopy type of X. In particular, one has an isomorphism of graded algebras $H^*(\wedge Z, d) \cong H^*(X; \mathbb{Q})$. The model is called minimal if $dZ \subset \wedge^{\geqslant 2} Z$. In this case the vector spaces Z^n and $\mathrm{Hom}(\pi_n(X), \mathbb{Q})$ are isomorphic. If X has the rational homotopy type of a finite CW-complex, we say that X is elliptic if Z is finite dimensional, otherwise X is called hyperbolic.

2. Models of classifying spaces

Henceforth X will denote a simply connected finite CW-complex and $\mathcal{L}_X$ its homotopy Lie algebra. Let $aut\,X$ denote the space of free self homotopy equivalences of X, $aut_1(X)$ the path component of $aut\,X$ containing the identity map of X. The space $Baut_1(X)$ classifies fibrations with fibre X over simply connected base spaces [4].

The Schlessinger-Stasheff model for $Baut_1(X)$ is defined as follows [**12**]. If $(\mathbb{L}(V), \delta)$ is a Quillen model of X, we define a differential Lie algebra $Der\mathbb{L}(V) = \oplus_{k \geqslant 1} \mathrm{Der}_k \mathbb{L}(V)$ where $\mathrm{Der}_k \mathbb{L}(V)$ is the vector space of derivations of $\mathbb{L}(V)$ which increase the degree by k, with the restriction that $\mathrm{Der}_1 \mathbb{L}(V)$ is the vector space of derivations of degree 1 that commute with the differential δ.

Define the differential Lie algebra $(s\mathbb{L}(V) \underset{\sim}{\oplus} \mathrm{Der}\,\mathbb{L}(V), D)$ as follows:

- The graded vector space $s\mathbb{L}(V) \underset{\sim}{\oplus} \mathrm{Der}\,\mathbb{L}(V)$ is isomorphic to $s\mathbb{L}(V) \oplus \mathrm{Der}\,\mathbb{L}(V)$,
- If $\theta, \gamma \in \mathrm{Der}\,\mathbb{L}(V)$ and $sx, sy \in s\mathbb{L}(V)$, then $[\theta, \gamma] = \theta\gamma - (-1)^{|\theta||\gamma|}\gamma\theta$, $[\theta, sx] = (-1)^{|\theta|} s\theta(x)$ and $[sx, sy] = 0$,
- The differential D is defined by $D\theta = [\delta, \theta]$, $D(sx) = -s\delta x + ad\,x$, where $ad\,x$ is the inner derivation determined by x.

From the Sullivan minimal model $(\wedge Z, d)$, Sullivan defines the graded differential Lie algebra $(\mathrm{Der} \wedge Z, D)$ as follows [**13**]. For $k > 1$, the vector space $(\mathrm{Der} \wedge Z)_k$ consists of the derivations on $\wedge Z$ that decrease the degree by k and $(\mathrm{Der} \wedge Z)_1$ is the vector space of derivations of degree 1 verifying $d\theta + \theta d = 0$. For $\theta, \gamma \in \mathrm{Der} \wedge V$, the Lie bracket is defined by $[\theta, \gamma] = \theta\gamma - (-1)^{|\theta||\gamma|}\gamma\theta$ and the differential D is defined by $D\theta = [d, \theta]$.

We have the following result:

Theorem 1. *[13, 12, 14] The differential Lie algebras* $(\mathrm{Der} \wedge Z, D)$ *and* $(s\mathbb{L}(V) \underset{\sim}{\oplus} \mathrm{Der}\,\mathbb{L}(V), D)$ *are models of the classifying space* $B\,aut_1(X)$.

An indirect proof of the Schlessinger-Stasheff model is given in [**8**, Theorem 2].

3. The classifying space spectral sequence

Recall that if (L, δ) is a graded differential Lie algebra, then L becomes an UL module by the adjoint representation $ad : L \to \mathrm{Hom}(L, L)$. In the sequel all Lie algebras are endowed with the above module structure.

Let $(\mathbb{L}(V), \delta)$ be a Quillen model of a finite CW-complex and (TV, d) its enveloping algebra. On the TV-module $TV \otimes (\mathbb{Q} \oplus sV)$, define a $\mathbb{Q}$-linear map

$$S : TV \otimes (\mathbb{Q} \oplus sV) \to TV \otimes (\mathbb{Q} \oplus sV)$$

as follows:

- $S(1 \otimes x) = 0$ for all $x \in \mathbb{Q} \oplus sV$,
- $S(v \otimes 1) = 1 \otimes sv$ for all $v \in V$,
- If $a \in TV$ and $x \in TV \otimes (\mathbb{Q} \oplus sV)$ with $|x| > 0$, then $S(a.x) = (-1)^{|a|} a.S(x)$.

The differential on the TV-module $TV \otimes (\mathbb{Q} \oplus sV)$ is defined by

$$D(1 \otimes sv) = v \otimes 1 - S(dv \otimes 1) \text{ for } v \in V \text{ and } D(1 \otimes 1) = 0.$$

It follows from [**1**] that $(TV \otimes (\mathbb{Q} \oplus sV), D)$ is acyclic, hence it is a semifree resolution of $\mathbb{Q}$ as a (TV, d)-module [**6**, §6].

Using the Schlessinger-Stasheff model of the classifying space, the author proved the following:

Theorem 2. *[8] The differential graded vector spaces $Hom_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$ and $s\mathbb{L}(V) \oplus \mathrm{Der}\,\mathbb{L}(V)$ are isomorphic. Moreover, for $n \geqslant 0$, the $\mathbb{Q}$-vector spaces $\mathrm{Ext}^n_{TV}(\mathbb{Q}, \mathbb{L}(V))$ and $\pi_{n+1}(\Omega B\, aut_1\, X) \otimes \mathbb{Q}$ are isomorphic.*

In particular if X is a coformal space, one has an isomorphism $\pi_n(B\, aut_1\, X) \otimes \mathbb{Q} \cong \mathrm{Ext}^n_{U\mathcal{L}_X}(\mathbb{Q}, \mathcal{L}_X)$. Therefore $\pi_*(B\, aut_1\, X) \otimes \mathbb{Q}$ can be computed by the means of a projective resolution of $\mathbb{Q}$ as an $U\mathcal{L}_X$-module.

Consider the complex $\mathcal{P} = Hom_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$. Filter V as follows

$$F_0 V = 0, \quad F_{p+1} V = \{x \in V : dx \in \mathbb{L}(F_p V)\}.$$

We will denote $V_p = F_p V / F_{p-1} V$. If $F_{n-1} V \neq F_n V = V$, following Lemaire [**10**] we say that V is of length n. We will restrict to spaces with a Quillen model of length n.

Define a filtration on $P = TV \otimes (\mathbb{Q} \oplus sV)$ as follows:

$$P_0 = TV \otimes \mathbb{Q},\ P_1 = TV \otimes (\mathbb{Q} \oplus sV_1), \ldots, P_n = TV \otimes (\mathbb{Q} \oplus sV_{\leqslant n}).$$

We filter the complex

$$\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$$

by

$$F_k = \{f : f(P_{k-1}) = 0\}.$$

This yields a spectral sequence E_r such that $E_1^{p,q} = \mathrm{Hom}_{\mathbb{Q}}(sV_p, \mathcal{L}_X)$ for $p > 1$, $E_1^{0,q} = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathcal{L}_X)$ and that converges to $\mathrm{Ext}^*_{TV}(\mathbb{Q}, \mathbb{L}(V))$. This sequence will be called the *classifying space spectral sequence* of X.

Now assume that X is coformal and let $A = U\mathcal{L}_X$. If $\mathbb{L}(V_1)/I$ is a minimal presentation of $\mathcal{L}_X$, then there is a quasi-isomorphism $(\mathbb{L}(V_1 \oplus V_2 \oplus \cdots \oplus V_n), \delta) \to \mathcal{L}_X$ which extends to $p : (TV, d) \xrightarrow{\simeq} (A, 0)$. The (E_1, d) term provides a resolution

$$\cdots \to A \otimes sV_n \to A \otimes sV_{n-1} \to \cdots \to A \otimes sV_1 \to A \to \mathbb{Q}$$

of $\mathbb{Q}$ as an A-module. Here the differential is given by the composition

$$sV_n \xrightarrow{D} TV \otimes (\mathbb{Q} \oplus sV_{n-1}) \xrightarrow{p \otimes id} A \otimes (\mathbb{Q} \oplus sV_{n-1}).$$

The spectral sequence will therefore collapse at E_2 level. Moreover $\mathrm{Ext}^*_A(\mathbb{Q}, \mathcal{L}_X)$ is endowed with a Lie algebra structure verifying

$$[\mathrm{Ext}^{p,*}, \mathrm{Ext}^{q,*}] \subset \mathrm{Ext}^{p+q-1,*}. \tag{2}$$

The Lie bracket can be defined using the bijection between the Koszul complex $C^*(\mathcal{L}_X, \mathcal{L}_X)$ and derivations on the Sullivan model $C^*(\mathcal{L}_X, \mathbb{Q})$ of X [9, Proposition 4] (see also [7] for a direct definition of the Lie bracket on $C^*(\mathcal{L}_X, \mathcal{L}_X)$). Alternatively one may use the bijection

$$\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V)) \cong s\mathbb{L}(V) \underset{\sim}{\oplus} \mathrm{Der}\,\mathbb{L}(V)$$

to transfer a Lie algebra structure on $\mathrm{Hom}_{TV}(TV \otimes (\mathbb{Q} \oplus sV), \mathbb{L}(V))$ from $s\mathbb{L}(V) \underset{\sim}{\oplus} \mathrm{Der}\,\mathbb{L}(V)$.

Definition 3. Let L be a Lie algebra. An element $x \in L$ is called locally nilpotent if for every $y \in L$, there is a positive integer k such that $(ad\,x)^k(y) = 0$. A subset $K \subset L$ is called locally nilpotent if each element of K is locally nilpotent.

We deduce from Equation (2) the following

Proposition 4. *Let X be a coformal space of homotopy Lie algebra denoted $\mathcal{L}_X$. If X has a Quillen model $(\mathbb{L}(V), \delta)$, of length n, one has:*

1. *For $k \neq 1$, $\mathrm{Ext}^k_A(\mathbb{Q}, \mathcal{L}_X)$ is locally nilpotent,*
2. *$\mathrm{Ext}^1_A(\mathbb{Q}, \mathcal{L}_X)$ is a subalgebra of $\mathrm{Ext}_A(\mathbb{Q}, \mathcal{L}_X)$,*
3. *If $\mathrm{Ext}^0_A(\mathbb{Q}, \mathcal{L}_X) = 0$, then $\oplus_{i \geqslant i_0} \mathrm{Ext}^i_A(\mathbb{Q}, \mathcal{L}_X)$ is an ideal of $\mathrm{Ext}_A(\mathbb{Q}, \mathcal{L}_X)$, for $i_0 \geqslant 1$.*

We will now assume that X is a coformal 2-cone. Recall that X has a Quillen minimal model of the form $(\mathbb{L}(V_1 \oplus V_2), \delta)$, with $\delta V_1 = 0$ and $\delta V_2 \subset \mathbb{L}(V_1)$. Moreover $\pi_*(\Omega X) \otimes \mathbb{Q} = H_*(\mathbb{L}(V_1 \oplus V_2), \delta) = \mathbb{L}(V_1)/I$, where I is the ideal of $\mathbb{L}(V_1)$ generated by δV_2.

Definition 5. Let $\mathbb{L}(V)$ be a free Lie algebra where $\{a, b, c, \dots\}$ is a basis of V. Denote $\mathbb{L}^n(V)$ the subspace of $\mathbb{L}(V)$ consisting of Lie brackets of length n. Consider a basis $\{u_1, u_2, \dots\}$ of $\mathbb{L}^n(V)$ where each u_i is a Lie monomial. If $x \in \{a, b, c, \dots\}$, we define the length of u_i in the variable x, $l_x(u_i)$, as the number of occurrences of the letter x in u_i. If $u = \sum r_i u_i \in \mathbb{L}^n(V)$, define $l_x(u) = \min\{l_x(u_i)\}$ and if $v = \sum v_i$ where $v_i \in \mathbb{L}^i(V)$, $l_x(v) = \min\{l_x(v_i)\}$.

It is straightforward that the above definition extends to the enveloping algebra $T(V)$.

Theorem 6. *Let X be a coformal 2-cone and $(\mathbb{L}(V_1 \oplus V_2), \delta)$ be its Quillen minimal model. Choose a basis $\{x_1, x_2, \dots\}$ for V_1 and a basis $\{y_1, y_2, \dots\}$ for V_2. If for some $x_k \in \{x_1, x_2, \dots\}$, $l_{x_k}(\delta y_j) \geqslant 2$ for all $y_j \in \{y_1, y_2, \dots\}$, then $\mathrm{Ext}^{2,*}_A(\mathbb{Q}, \mathcal{L}_X)$ is infinite dimensional.*

Proof. Note that for $i \neq k$ the element $(ad\, x_i)^n(x_k)$ is a nonzero homology class in $H_*(\mathbb{L}(V_1 \oplus V_2), \delta)$ as it contains only one occurrence of x_k. Take $y_t \in \{y_1, y_2, \dots\}$ and $x_m \in \{x_1, x_2, \dots\}$ with $m \neq k$. For each $n \geqslant 1$, define $f_n \in \mathrm{Hom}_A(A \otimes sV_2, \mathcal{L}_X)$ by $f_n(sy_t) = (ad\, x_m)^n(x_k)$ and $f_n(sy_j) = 0$ for $j \neq t$. Obviously $f_n \in \mathrm{Hom}_A(A \otimes sV_2, \mathcal{L}_X)$ is a cocycle. Suppose that f_n is a coboundary. There exists $g_n \in \mathrm{Hom}_A(A \otimes sV_1, \mathcal{L}_X)$ such that $f_n(sy_t) = g_n(dsy_t)$. From the definition of the differential d, one has $dsy_t = \sum_i p_i s x_i$, where the p_i's are polynomials in the variables $x_1, x_2, \dots$. From the hypothesis on the differential dy_t one knows that $l_{x_k}(p_i) \geqslant 2$ for $i \neq k$ and $l_{x_k}(p_k) \geqslant 1$. By using the number of occurrences of the variable x_k, one deduces from the previous equalities that $(ad\, x_m)^n(x_k)$ equals the component of length 1 in x_k of $p_k g_n(sx_k)$. Therefore, in the monomial decomposition of $g_n(sx_k)$ (resp. p_k) there must exist $(ad\ x_m)^{n-s}(x_k)$ (resp. x_m^s). We obtain a contradiction with $l_{x_k}(p_k) \geqslant 1$.

The cocycles f_n create an infinite number of non zero classes (of distinct degrees) and the space $\mathrm{Ext}_A^{2,*}(\mathbb{Q}, \mathcal{L}_X)$ is infinite dimensional. □

Corollary 7. *If hypotheses of the above theorem are satisfied, then* $\mathrm{cat}(B\, aut_1(X)) = \infty$.

Proof. If $sx \in \mathrm{Ext}^{0,*} \subset \mathbb{L}(V_1)/I$ and $f \in \mathrm{Ext}^{2,*}$ then $[f, sx] = \pm sf(x)$. As elements of $\mathrm{Ext}^{2,*}$ vanish on V_1, we deduce that $[\mathrm{Ext}^{2,*}, \mathrm{Ext}^{0,*}] = 0$. It follows from Theorem 6 that $J = \mathrm{Ext}^2_{U\mathcal{L}_X}(\mathbb{Q}, \mathcal{L}_X)$ is an infinite dimensional ideal of $\pi_*(\Omega B\, aut_1(X))$. Moreover it follows from Equation (2) that J is abelian. We deduce that the category of $B\, aut_1(X)$ is infinite [**5**, Theorem 12.2]. □

If X is an elliptic space of Sullivan minimal model $(\wedge Z, d)$ then $Der \wedge Z$ is a finite dimensional $\mathbb{Q}$-vector space. Hence the homotopy Lie algebra of $B\, aut_1(X)$ is finite dimensional, therefore $\pi_*(\Omega B\, aut_1(X)) \otimes \mathbb{Q}$ is nilpotent. In [**11**], P. Salvatore proved that if $X = S^{2n+1} \vee S^{2n+1}$, then $\pi_*(\Omega B\, aut_1(X)) \otimes \mathbb{Q}$ contains an element α that is not locally nilpotent. The proof consists in the construction of two outer derivations α and β of the free Lie algebra $\mathbb{L}(a, b)$, where $|a| = |b| = 2n$, such that $(ad\, \alpha)^i(\beta) \neq 0$, for every integer $i > 0$. The technique can be applied to any free Lie algebra with at least two generators. Therefore $\pi_*(\Omega B\, aut_1(X)) \otimes \mathbb{Q}$ contains an element α that is not locally nilpotent if X is a wedge of two spheres or more.

P. Salvatore asked if $\pi_*(\Omega B\, aut_1(X)) \otimes \mathbb{Q}$ has always such a property for every hyperbolic space X. A positive answer to this question would provide another characterization of the elliptic-hyperbolic dichotomy [**5**].

For a product space we have the following

Proposition 8. *If $X = Y \times Z$ is a product space such that the Lie algebra $\pi_*(\Omega B\, aut_1(Y)) \otimes \mathbb{Q}$ is not nilpotent, then $\pi_*(\Omega B\, aut_1(X)) \otimes \mathbb{Q}$ is not nilpotent.*

Proof. Let $(\wedge V, d)$ and $(\wedge W, d')$ be Sullivan models of Y and Z respectively. Therefore $(\wedge V \otimes \wedge W, d \otimes d')$ is a Sullivan model of X. It follows from [**12**] that

$$H_*(Der(\wedge V \otimes \wedge W)) \cong H_*(Der \wedge V) \otimes H^*(\wedge W) \oplus H^*(\wedge V) \otimes H_*(Der \wedge W).$$

Therefore $\pi_*(\Omega B\, aut_1(Y)) \otimes \mathbb{Q}$ is a subalgebra of $\pi_*(\Omega B\, aut_1(X)) \otimes \mathbb{Q}$. □

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