On semi-parallel lightlike hypersurfaces of indefinite Kenmotsu manifolds

Fortuné Massamba

Abstract. We study semi-parallel lightlike hypersurfaces of an indefinite Kenmotsu manifold, tangent to the structure vector field. Some Theorems on parallel and semi-parallel vector field, geodesibility of lightlike hypersurfaces are obtained. The geometrical configuration of such lightlike hypersurfaces is established. We prove that, in totally contact umbilical lightlike hypersurfaces of an indefinite Kenmotsu manifold which has constant $\overline{\phi}$-holomorphic sectional curvature c, tangent to the structure vector field and such that its distribution is parallel, the parallelism and semi-parallelism notions are equivalent.

Mathematics Subject Classification (2010). Primary 53C15; Secondary 53C25, 53C40, 53C50.

Keywords. Semi-parallel, lightlike hypersurfaces, indefinite Kenmotsu space form, screen distribution.

1. Introduction

A semi-Riemannian manifold is *locally symmetric* if it satisfies the condition $\nabla R = 0$, where ∇ is the Levi–Civita connection on semi-Riemannian manifold and R is the corresponding curvature tensor. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$. Manifolds which satisfy the latter condition are called *semi-symmetric* and have been classified, in Riemannian case, by Szabo in [14] and [15].

In the theory of submanifolds of a space form, conditions analogous to local symmetry and semi-symmetry have been introduced and studied quite intensively. Ferus [8] and others introduced the concept of parallel immersions, that is, immersions with parallel second fundamental form, and classified such immersions. On the other hand, Deprez and others introduced and studied the concept of semi-parallel immersions, that is, immersions such that the curvature tensor annihilates the second fundamental form (for details, see [1,5,6] and references therein).

The present paper aims to investigate semi-parallel lightlike hypersurfaces of indefinite Kenmotsu manifolds.

As is well known, contrary to timelike and spacelike hypersurfaces, the geometry of a lightlike hypersurface is different and rather difficult since the normal bundle and the tangent bundle have non-zero intersection. To overcome this difficulty, a theory on the differential geometry of lightlike hypersurfaces developed by Duggal and Bejancu [7] introduces a non-degenerate screen distribution and constructs the corresponding lightlike transversal vector bundle. This enables to define an induced linear connection (depending on the screen distribution, and hence is not unique in general).

The paper is organized as follows. In Sect. 2, we recall some basic definitions and formulas for indefinite Kenmotsu manifolds supported by an example and lightlike hypersurfaces of semi-Riemannian manifolds. In Sect. 3, theorems on parallel and semi-parallel vector field, geodesibility of lightlike hypersurfaces of indefinite Kenmotsu manifolds are obtained. We also study totally contact umbilical lightlike hypersurfaces of an indefinite Kenmotsu manifold. By Theorem 3.6, we establish the geometrical configuration of such lightlike hypersurfaces, tangent to the structure vector field in Kenmotsu manifolds which have constant $\overline{\phi}$-holomorphic sectional curvature c. We prove the non-existence of totally contact umbilical lightlike hypersurfaces, tangent to the structure vector field, in indefinite Kenmotsu manifold under a certain condition. We show that, in totally contact umbilical lightlike hypersurfaces of indefinite Kenmotsu manifolds which have constant $\overline{\phi}$-holomorphic sectional curvature c, tangent to the structure vector field and such that its distribution is parallel, the parallelism and semi-parallelism notions are equivalent (Theorem 3.13). Finally, we discuss the effect of the change of the screen distribution on different results found.

2. Preliminaries

Let $\overline{M}$ be a $(2n+1)$-dimensional manifold endowed with an almost contact structure $(\overline{\phi}, \xi, \eta)$, i.e. $\overline{\phi}$ is a tensor field of type $(1, 1)$, ξ is a vector field, and η is a 1-form satisfying

$$\overline{\phi}^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \overline{\phi} = 0 \quad \text{and} \quad \overline{\phi}\xi = 0. \tag{2.1}$$

Then $(\overline{\phi}, \xi, \eta, \overline{g})$ is called an almost contact metric structure on $\overline{M}$ if $(\overline{\phi}, \xi, \eta)$ is an almost contact structure on $\overline{M}$ and $\overline{g}$ is a semi-Riemannian metric on $\overline{M}$ such that, for any vector field $\overline{X}, \overline{Y}$ on $\overline{M}$ [3]

$$\eta(\overline{X}) = \overline{g}(\xi, \overline{X}), \quad \overline{g}(\overline{\phi X}, \overline{\phi Y}) = \overline{g}(\overline{X}, \overline{Y}) - \eta(\overline{X})\eta(\overline{Y}). \tag{2.2}$$

If, moreover, $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi X}$ and $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$, where $\overline{\nabla}$ is the Levi–Civita connection for the semi-Riemannian metric $\overline{g}$, we call $\overline{M}$ an indefinite Kenmotsu manifold [10].

A plane section σ in $T_p\overline{M}$ is called a $\overline{\phi}$-section if it is spanned by $\overline{X}$ and $\overline{\phi X}$, where $\overline{X}$ is a unit tangent vector field orthogonal to ξ. Since $\overline{\phi}\sigma=\sigma$, the $\overline{\phi}$-section σ is a holomorphic $\overline{\phi}$-section and the sectional curvature of a $\overline{\phi}$-section σ is called a $\overline{\phi}$-holomorphic sectional curvature (see [4]), [9] and references therein for more details). If a Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature c, then, by virtue of the Proposition 12 in [10], the curvature tensor $\overline{R}$ of $\overline{M}$ is given by, for any $\overline{X}$, $\overline{Y}$, $\overline{Z}\in\Gamma(T\overline{M})$,

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y})\overline{Z} &= \frac{c-3}{4}\left\{\overline{g}(\overline{Y},\overline{Z})\overline{X}-\overline{g}(\overline{X},\overline{Z})\overline{Y}\right\}+\frac{c+1}{4}\left\{\eta(\overline{X})\eta(\overline{Z})\overline{Y}\right.\\ &\quad-\eta(\overline{Y})\eta(\overline{Z})\overline{X}+\overline{g}(\overline{X},\overline{Z})\eta(\overline{Y})\xi-\overline{g}(\overline{Y},\overline{Z})\eta(\overline{X})\xi+\overline{g}(\overline{\phi Y},\overline{Z})\overline{\phi X}\\ &\quad\left.-\,\overline{g}(\overline{\phi X},\overline{Z})\overline{\phi Y}-2\overline{g}(\overline{\phi X},\overline{Y})\overline{\phi Z}\right\}. \qquad (2.3)\end{aligned}$$

A Kenmotsu manifold is a typical example of $C(\alpha)$-manifold, with $\alpha=-1$, introduced by Janssens and Vanhecke [9].

Note that the $\overline{\phi}$-holomorphic sectional curvature of an indefinite $C(\alpha)$-manifold does not satisfy, in general, a "Schur Lemma" although it holds for co-Kähler and indefinite Sasakian manifolds (see [4] for details).

An indefinite Kenmotsu manifold which has constant $\overline{\phi}$-holomorphic sectional curvature c will be denoted by $\overline{M}^c$. A Kenmotsu manifold $\overline{M}$ of constant $\overline{\phi}$-holomorphic sectional curvature c will be called *Kenmotsu space form* and denoted by $\overline{M}(c)$. Here $\overline{M}^c$ is different from $\overline{M}(c)$ and this is well specified in [10] through Proposition 12 and Theorem 13.

Example 1. We consider the 7-dimensional manifold $\overline{M}=\{(x_1,x_2,\ldots,x_7)\in\mathbb{R}^7 : x_7\neq 0\}$, where $x=(x_1,x_2,\ldots,x_7)$ are the standard coordinates in $\mathbb{R}^7$. The vector fields

$$\begin{aligned}&e_1=x_7\frac{\partial}{\partial x_1},\quad e_2=x_7\frac{\partial}{\partial x_2},\quad e_3=x_7\frac{\partial}{\partial x_3},\quad e_4=x_7\frac{\partial}{\partial x_4},\quad e_5=-x_7\frac{\partial}{\partial x_5},\\ &e_6=-x_7\frac{\partial}{\partial x_6},\quad e_7=-x_7\frac{\partial}{\partial x_7},\end{aligned}$$

are linearly independent at each point of $\overline{M}$. Let $\overline{g}$ be the semi-Riemannian metric defined by $\overline{g}(e_i,e_j)=0$, $\forall i\neq j$, $i,j=1,2,\ldots,7$ and $\overline{g}(e_k,e_k)=1$, $\forall\, k=1,2,3,4,7$, $\overline{g}(e_m,e_m)=-1$, $\forall m=5,6$. That is, the form of the metric becomes

$$\overline{g}=\frac{1}{(x_7)^2}\left\{dx_1^2+dx_2^2+dx_3^2+dx_4^2-dx_5^2-dx_6^2+dx_7^2\right\}.$$

Let η be the 1-form defined by $\eta(\overline{X})=\overline{g}(\overline{X},e_7)$, for any $\overline{X}\in\Gamma(T\overline{M})$.

Let $\overline{\phi}$ be the $(1,1)$ tensor field defined by $\overline{\phi}e_1=-e_2$, $\overline{\phi}e_2=e_1$, $\overline{\phi}e_3=-e_4$, $\overline{\phi}e_4=e_3$, $\overline{\phi}e_5=-e_6$, $\overline{\phi}e_6=e_5$, $\overline{\phi}e_7=0$. Then using the linearity of $\overline{\phi}$ and $\overline{g}$, we have $\eta(e_7)=1$, $\overline{\phi}^2\overline{X}=-\overline{X}+\eta(\overline{X})e_7$, $\overline{g}(\overline{\phi}\,\overline{X},\overline{\phi}\overline{Y})=\overline{g}(\overline{X},\overline{Y})-\eta(\overline{X})\eta(\overline{Y})$, for any $\overline{X}$, $\overline{Y}\in\Gamma(T\overline{M})$. Thus, for $e_7=\xi$, $(\overline{\phi},\xi,\eta,\overline{g})$ defines an almost contact metric structure on $\overline{M}$. Let $\overline{\nabla}$ be the Levi–Civita connection with respect to the metric $\overline{g}$. Then, we have $[e_i,e_7]=e_i$, $\forall i=1,2,\ldots,6$ and

$[e_i, e_j] = 0, \forall\, i \neq j,\, i,\, j = 1, 2, \ldots, 6$. The metric connection $\overline{\nabla}$ of the metric $\overline{g}$ is given by

$$2\overline{g}(\nabla_{\overline{X}}\overline{Y}, \overline{Z}) = \overline{X}.\overline{g}(\overline{Y}, \overline{Z}) + \overline{Y}.\overline{g}(\overline{Z}, \overline{X}) - \overline{Z}.\overline{g}(\overline{X}, \overline{Y}) - \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}])$$
$$-\overline{g}(\overline{Y}, [\overline{X}, \overline{Z}]) + \overline{g}(\overline{Z}, [\overline{X}, \overline{Y}]),$$

which is known as Koszul's formula. Using this formula, the non-vanishing covariant derivatives are given by

$$\overline{\nabla}_{e_1}e_1 = -e_7, \quad \overline{\nabla}_{e_2}e_2 = -e_7, \quad \overline{\nabla}_{e_3}e_3 = -e_7, \quad \overline{\nabla}_{e_4}e_4 = -e_7,$$
$$\overline{\nabla}_{e_5}e_5 = e_7, \quad \overline{\nabla}_{e_6}e_6 = e_7, \quad \overline{\nabla}_{e_1}e_7 = e_1, \quad \overline{\nabla}_{e_2}e_7 = e_2,$$
$$\overline{\nabla}_{e_3}e_7 = e_3, \quad \overline{\nabla}_{e_4}e_7 = e_4, \quad \overline{\nabla}_{e_5}e_7 = e_5, \quad \overline{\nabla}_{e_6}e_7 = e_6.$$

From these relations, it follows that the manifold $\overline{M}$ satisfies $(\overline{\nabla}_{\overline{X}}\overline{\phi})\overline{Y} = \overline{g}(\overline{\phi X}, \overline{Y})\xi - \eta(\overline{Y})\overline{\phi X}$ and $\overline{\nabla}_{\overline{X}}\xi = \overline{X} - \eta(\overline{X})\xi$. Hence, $\overline{M}$ is an indefinite Kenmotsu manifold.

Let $(\overline{M}, \overline{g})$ be a $(2n+1)$-dimensional semi-Riemannian manifold with index s, $0 < s < 2n+1$ and let (M, g) be a hypersurface of $\overline{M}$, with $g = \overline{g}_{|M}$. M is said to be a lightlike hypersurface of $\overline{M}$ if the metric g is of constant rank $2n-1$ and the orthogonal complement $TM^{\perp}$ of tangent space TM, defined as

$$TM^{\perp} = \bigcup_{p \in M} \left\{ Y_p \in T_p\overline{M} : \overline{g}_p(X_p, Y_p) = 0, \ \forall X_p \in T_pM \right\}, \tag{2.4}$$

is a distribution of rank 1 on M [7]: $TM^{\perp} \subset TM$ and then coincides with the radical distribution Rad$TM = TM \cap TM^{\perp}$. A complementary bundle of $TM^{\perp}$ in TM is a constant rank $2n-1$ non-degenerate distribution over M. It is called a *screen distribution* and is often denoted by $S(TM)$. Existence of $S(TM)$ is secured provided M is paracompact. However, in general, $S(TM)$ is not canonical (thus it is not unique) and the lightlike geometry depends on its choice but it is canonically isomorphic to the vector bundle $TM/$RadTM [12].

A lightlike hypersurface endowed with a specific screen distribution is denoted by the triple $(M, g, S(TM))$. As $TM^{\perp}$ lies in the tangent bundle, the following result has an important role in studying the geometry of a lightlike hypersurface [7].

Theorem 2.1 (Duggal–Bejancu). *Let $(M, g, S(TM))$ be a lightlike hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $N(TM)$ of rank 1 over M such that for any non-zero section E of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exist a unique section N of $N(TM)$ on $\mathcal{U}$ satisfying*

$$\overline{g}(N, E) = 1 \quad \textit{and} \quad \overline{g}(N, N) = \overline{g}(N, W) = 0, \quad \forall\, W \in \Gamma(S(TM)|_{\mathcal{U}}).$$

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by Γ(E) the smooth sections of the vector bundle E. Also by $\perp$ and $\oplus$ we denote the orthogonal and nonorthogonal direct sum of two vector bundles. By Theorem 2.1 we may write down the following decompositions

$$\begin{aligned} TM &= S(TM) \perp TM^{\perp}, \\ T\overline{M} &= TM \oplus N(TM) = S(TM) \perp (TM^{\perp} \oplus N(TM)). \end{aligned} \tag{2.5}$$

Let $\overline{\nabla}$ be the Levi–Civita connection on $(\overline{M}, \overline{g})$, then by using the second decomposition of (2.5) and considering a normalizing pair $\{E, N\}$ as in Theorem 2.1, we have Gauss and Weingarten formulae in the form

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}), \tag{2.6}$$

$$\text{and} \quad \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}),\ V \in \Gamma(N(TM)), \tag{2.7}$$

where $\nabla_X Y$, $A_V X \in \Gamma(TM)$ and $h(X, Y)$, $\nabla^{\perp}_X V \in \Gamma(N(TM))$. ∇ is an induced symmetric linear connection on M, $\nabla^{\perp}$ is a linear connection on the vector bundle $N(TM)$, h is a $\Gamma(N(TM))$-valued symmetric bilinear form and A_V is the shape operator of M concerning V.

Equivalently, consider a normalizing pair $\{E, N\}$ as in Theorem 2.1. Then (2.6) and (2.7) take the form

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}) \tag{2.8}$$

$$\text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.9}$$

where B, A_N, τ and ∇ are called the local second fundamental form, the local shape operator, the transversal differential 1-form and the induced linear torsion free connection, respectively, on $TM_{|\mathcal{U}}$.

It is important to mention that the local second fundamental form B of M is independent of the choice of screen distribution. In fact, from (2.8) and (2.9), we obtain $B(X, Y) = \overline{g}(\overline{\nabla}_X Y, E)$ and $\tau(X) = \overline{g}(\nabla^{\perp}_X N, E)$.

Let P be the projection morphism of TM on $S(TM)$ with respect to the orthogonal decomposition of TM. We have

$$\nabla_X PY = \nabla^{*}_X PY + C(X, PY)E, \quad \forall X, Y \in \Gamma(TM|_{\mathcal{U}}) \tag{2.10}$$

$$\text{and} \quad \nabla_X E = -A^{*}_E X - \tau(X)E, \quad \forall X \in \Gamma(TM|_{\mathcal{U}}), \tag{2.11}$$

where $\nabla^{*}_X PY$ and $A^{*}_E X$ belong to $\Gamma(S(TM))$. C, A^{*}_E and ∇^{*} are called the local second fundamental form, the local shape operator and the induced connection, respectively, on $S(TM)$. The induced linear connection ∇ is not a metric connection and we have, for any X, $Y \in \Gamma(TM|_{\mathcal{U}})$,

$$(\nabla_X g)(Y, Z) = B(X, Y)\theta(Z) + B(X, Z)\theta(Y), \tag{2.12}$$

where θ is a differential 1-form locally defined on M by $\theta(\cdot) := \overline{g}(N, \cdot)$.

Also, we have the following identities,

$$g(A^{*}_E X, PY) = B(X, PY), \quad g(A^{*}_E X, N) = 0, \quad B(X, E) = 0. \tag{2.13}$$

Finally, using (2.8) and (2.9), $\overline{R}$ and R are the curvature tensor fields of $\overline{M}$ and M are related as

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X \\ &\quad + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) \\ &\quad - \tau(Y)B(X,Z)\}\, N, \qquad (2.14)\end{aligned}$$

$$\text{where} \quad (\nabla_X B)(Y,Z) = X.B(Y,Z) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z). \qquad (2.15)$$

Suppose that the distribution $S(TM)$ is parallel with respect to ∇, that is, $\nabla_X PY \in \Gamma(S(TM))$ for any X, $Y \in \Gamma(TM)$. As ∇ is a torsion-free connection, it follows that $S(TM)$ is integrable. Moreover, from (2.10) and (2.13), we have the following result proved in [7].

THEOREM 2.2 (Duggal–Bejancu). *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, the following assertions are equivalent:*

(i) *$S(TM)$ is parallel with respect to the induced connection ∇.*
(ii) *C vanishes identically on M.*
(iii) *A_N vanishes identically on M.*

3. Main results

Let $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ be an indefinite Kenmotsu manifold and let (M, g) be a lightlike hypersurface, *tangent to the structure vector field ξ ($\xi \in TM$)* of $(\overline{M}, \overline{g})$. If E is a local section of $TM^{\perp}$, then by using the second equation of (2.2), one obtains $\overline{g}(\overline{\phi}E, E) = -\overline{g}(E, \overline{\phi}E)$, that is, $\overline{g}(\overline{\phi}E, E) = 0$, and by (2.4), $\overline{\phi}E$ is tangent to M. Thus $\overline{\phi}(TM^{\perp})$ is a distribution on M of rank 1 such that $\overline{\phi}(TM^{\perp}) \cap TM^{\perp} = \{0\}$. This enables us to choose a screen distribution $S(TM)$ such that it contains $\overline{\phi}(TM^{\perp})$ as a vector subbundle. If we consider a local section N of $N(TM)$, since $\overline{g}(\overline{\phi}N, E) = -\overline{g}(N, \overline{\phi}E) = 0$, we deduce that $\overline{\phi}E$ belongs to $S(TM)$. On the other hand, since $\overline{g}(\overline{\phi}N, N) = 0$, we see that the component of $\overline{\phi}N$ with respect to E vanishes. Thus $\overline{\phi}N \in \Gamma(S(TM))$.

From (2.2) and Theorem 2.1, we have $\overline{g}(\overline{\phi}N, \overline{\phi}E) = 1$. Therefore, $\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))$ (direct sum but not orthogonal) is a nondegenerate vector subbundle of $S(TM)$ of rank 2.

If M is tangent to the structure vector field ξ, then, we may choose $S(TM)$ so that ξ *belongs to* $S(TM)$. Using this, and since $\overline{g}(\overline{\phi}E, \xi) = \overline{g}(\overline{\phi}N, \xi) = 0$, there exists a nondegenerate distribution D_0 of rank $2n - 4$ on M such that

$$S(TM) = \left\{\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM))\right\} \perp D_0 \perp \langle\xi\rangle, \qquad (3.1)$$

where $\langle\xi\rangle$ is the distribution spanned by ξ, that is, $\langle\xi\rangle = Span\{\xi\}$. It is easy to check that the distribution D_0 is invariant under $\overline{\phi}$, i.e. $\overline{\phi}(D_0) = D_0$.

Example 2. Let M be a hypersurface of $(\overline{M}, \overline{\phi}, \xi, \eta, \overline{g})$ (indefinite Kenmotsu manifold defined in Example 1) given by

$$x_5 = \sqrt{2}\,(x_2 + x_3)\,,$$

where $(x_1, x_2, \dots, x_7)$ is a local coordinate system for $\mathbb{R}^7$. Thus, the tangent space TM is spanned by $\{U_i\}_{1\le i\le 6}$, where $U_1 = e_1$, $U_2 = e_2 - e_3$, $U_3 = \frac{1}{\sqrt{2}}(e_2 + e_3) - e_5$, $U_4 = e_4$, $U_5 = e_6$, $U_6 = \xi$ and the 1-dimensional distribution $TM^\perp$ of rank 1 is spanned by E, where $E = \frac{1}{\sqrt{2}}(e_2 + e_3) - e_5$. It follows that $TM^\perp \subset TM$. Then M is a 6-dimensional lightlike hypersurface of $\overline{M}$. Also, the transversal bundle $N(TM)$ is spanned by $N = \frac{1}{2}\{\frac{1}{\sqrt{2}}(e_2 + e_3) + e_5\}$. On the other hand, by using the almost contact structure of $\overline{M}$ and also by taking into account the decomposition (3.1), the distribution D_0 is spanned by $\{F, \overline{\phi}F\}$, where $F = U_2, \overline{\phi}F = U_1 + U_4$ and the distributions $\langle\xi\rangle$, $\overline{\phi}(TM^\perp)$ and $\overline{\phi}(N(TM))$ are spanned, respectively, by $\xi, \overline{\phi}E = \frac{1}{\sqrt{2}}(U_1 - U_4) + U_5$ and $\overline{\phi}N = \frac{1}{2}\{\frac{1}{\sqrt{2}}(U_1 - U_4) - U_5\}$. Hence, M is a lightlike hypersurface of $\overline{M}$.

Moreover, from (2.5) and (3.1) we obtain the decompositions

$$TM = \{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\} \perp D_0 \perp \langle\xi\rangle \perp TM^\perp, \tag{3.2}$$

$$T\overline{M} = \{\overline{\phi}(TM^\perp) \oplus \overline{\phi}(N(TM))\} \perp D_0 \perp \langle\xi\rangle \perp (TM^\perp \oplus N(TM)). \tag{3.3}$$

Now, we consider the distributions on M,

$$D := TM^\perp \perp \overline{\phi}(TM^\perp) \perp D_0 \quad \text{and} \quad D' := \overline{\phi}(N(TM)).$$

Then D is invariant under $\overline{\phi}$ and

$$TM = (D \oplus D') \perp \langle\xi\rangle. \tag{3.4}$$

Let us consider the local lightlike vector fields $U := -\overline{\phi}N$ and $V := -\overline{\phi}E$. Then, from (3.4), any $X \in \Gamma(TM)$ is written as

$$X = RX + QX + \eta(X)\xi, \quad QX = u(X)U,$$

where R and Q are the projection morphisms of TM into D and D', respectively, and u is a differential 1-form on M locally defined by $u(\cdot) := g(V, \cdot)$.

Applying $\overline{\phi}$ to X and (2.1) (note that $\overline{\phi}^2 N = -N$), we obtain $\overline{\phi}X = \phi X + u(X)N$, where ϕ is a tensor field of type $(1, 1)$ on M defined by $\phi X := \overline{\phi}RX$ and we also have

$$\phi^2 X = -X + \eta(X)\xi + u(X)U, \quad \forall\, X \in \Gamma(TM). \tag{3.5}$$

By using (2.1) we derive $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) - u(Y)v(X) - u(X)v(Y)$, where v is a differential 1-form on M locally defined by $v(\cdot) := g(U, \cdot)$. We note that

$$g(\phi X, Y) + g(X, \phi Y) = -u(X)\theta(Y) - u(Y)\theta(X). \tag{3.6}$$

We have the following useful identities: for any $X, Y \in \Gamma(TM)$,

$$\nabla_X \xi = X - \eta(X)\xi, \tag{3.7}$$
$$B(X,\xi) = 0, \tag{3.8}$$
$$C(X,\xi) = \theta(X), \tag{3.9}$$
$$B(X,U) = C(X,V). \tag{3.10}$$

Next, we give a characterization on parallel lightlike hypersufaces of an indefinite Kenmotsu manifold. In fact, it shows that there do not exist non-totally geodesic totally umbilical lightlike hypersurfaces of indefinite Kenmotsu manifolds, tangent to the structure vector field ξ.

If a $(2n+1)$-dimensional indefinite Kenmotsu manifold $\overline{M}$ has constant $\overline{\phi}$-holomorphic sectional curvature c, then, by virtue of Proposition 12 [10], the Ricci tensor $\overline{Ric}$ and the scalar curvature $\overline{r}$ are given, respectively, by [10]

$$\overline{Ric} = \frac{1}{2}\left(n(c-3)+c+1\right)\overline{g} - \frac{1}{2}(n+1)(c+1)\eta\otimes\eta, \tag{3.11}$$
$$\overline{r} = \frac{1}{2}\left(n(2n+1)(c-3) - n(c+1)\right). \tag{3.12}$$

This means that $\overline{M}^c$ is η-Einstein. But if $\overline{M}$ becomes a space of constant $\overline{\phi}$-holomorphic sectional curvature c, that is, a Kenmotsu space form $\overline{M}(c)$, the curvature tensor of $\overline{M}(c)$ has also the form given in (2.3) with c constant which implies, through the Eq. (3.11), that $\overline{M}(c)$ is η-Einstein. Since the coefficients of $\overline{Ric}$ are constant on $\overline{M}(c)$, by Corollary 9 in [10], $\overline{M}$ is an Einstein one and consequently, $c+1=0$, that is, $c=-1$. So, the Ricci tensor (3.11) becomes $\overline{Ric} = -2n\overline{g}$ and the scalar curvature is given by $\overline{r} = -2n(2n+1)$.

Thus, *if a Kenmotsu manifold $\overline{M}$ is a space form, then it is Einstein and* $c=-1$. This means that it is a space of constant curvature -1, so, locally it is isometric to the hyperbolic space.

Let M be a lightlike hypersurface of $\overline{M}^c$. Let us consider the pair $\{E, N\}$ on $\mathcal{U} \subset M$ (see Theorem 2.1) and by using (2.14), we obtain

$$\begin{aligned}(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) &= \tau(Y)B(X,Z) - \tau(X)B(Y,Z)\\ &+ \frac{c+1}{4}\left\{\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\right\}.\end{aligned} \tag{3.13}$$

The second fundamental form $h = B\otimes N$ of M is said to be parallel if

$$(\nabla_X h)(Y,Z) = 0, \tag{3.14}$$

for any $X, Y, Z \in \Gamma(TM)$. That is,

$$(\nabla_X B)(Y,Z) = -\tau(X)B(Y,Z). \tag{3.15}$$

This means that, in general, the parallelism of h does not imply the parallelism of B and vice versa.

Lemma 3.1. *There exist no lightlike hypersufaces of indefinite Kenmotsu manifolds $(\overline{M}^c, c \neq -1)$ with $\xi \in TM$ and parallel second fundamental form.*

Proof. Suppose $c \neq -1$ and second fundamental form is parallel. Then, if we take $Y = E$ and $Z = U$ in (3.13), we obtain $\frac{c+1}{4}u(X) = 0$. Taking $X = U$, we have $c = -1$, which is a contradiction. □

A lightlike hypersurface M is totally geodesic (respectively $D \perp \langle\xi\rangle$ or D'-totally geodesic) if the local second fundamental form B satisfies $B(X,Y) = 0$, for any $X, Y \in \Gamma(TM)$ (respectively $X, Y \in \Gamma(D \perp \langle\xi\rangle)$ or $\Gamma(D')$).

Lemma 3.2. *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$, such that its local second fundamental form B is parallel. If $\tau(E) \neq 0$, then $c = -1$ if and only if M is D'-totally geodesic.*

Proof. Suppose B is parallel. Then, $\nabla B = 0$ on M. Using this, the relation (3.13) becomes, for any $X, Y, Z \in \Gamma(TM)$,

$$\tau(X)B(Y,Z) - \tau(Y)B(X,Z) = \frac{c+1}{4}\left\{\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}X,Y)u(Z)\right\}. \tag{3.16}$$

Taking $Y = E$ in (3.16), we obtain

$$3\frac{c+1}{4}u(X)u(Z) = \tau(E)B(X,Z). \tag{3.17}$$

Taking $X = Z = U$ in (3.17), we have $3\frac{c+1}{4} = \tau(E)B(U,U)$ and if $\tau(E) \neq 0$, the equivalence follows. □

Theorem 3.3. *Let M be a lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ with $\xi \in TM$. If the second fundamental form h of M is parallel, then M is totally geodesic.*

Proof. Suppose that the second fundamental form h of M is parallel. Then (3.15) is satisfied. Taking $Z = \xi$ in (3.15) and using (3.8), we obtain

$$(\nabla_X B)(Y,\xi) = -\tau(X)B(Y,\xi) = 0. \tag{3.18}$$

From (2.15) and using (3.7) and (3.8), the left hand side of (3.18) becomes

$$(\nabla_X B)(Y,\xi) = -B(X,Y). \tag{3.19}$$

From the expressions (3.18) and (3.19) we complete the proof. □

We note that the Theorem 3.3 arises when the local second fundamental form B of M is also parallel.

A submanifold of a semi-Riemannian manifold with parallel second fundamental form h is called a parallel submanifold. So, the Theorem 3.3 generates some lightlike geometric aspects on any parallel lightlike hypersurface of an indefinite Kenmotsu manifold by using, for instance, the Duggal–Bejancu Theorem [7, Theorem 2.2, p. 88].

A submanifold M is said to be a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$ if the local second fundamental form B of M satisfies

$$B(X,Y) = \rho g(X,Y), \quad \forall X,Y \in \Gamma(TM) \tag{3.20}$$

where ρ is a smooth function on $\mathcal{U} \subset M$.

If we assume that M is a totally umbilical lightlike hypersurface of a semi-Riemannian manifold $\overline{M}$, then we have $B(X,Y) = \rho g(X,Y)$, for any $X,Y \in \Gamma(TM)$, which implies, by using (3.8), that $0 = B(\xi,\xi) = \rho$. Hence M is totally geodesic. Therefore we have

Proposition 3.4. *Let (M,g) be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M},\overline{g})$ with $\xi \in TM$. If M is totally umbilical, then M is totally geodesic.*

It follows from Proposition 3.4 that a Kenmotsu manifold $\overline{M}$ does not admit any non-totally geodesic, totally umbilical lightlike hypersurface. From this point of view, Bejancu [2] considered the concept of totally contact umbilical semi-invariant submanifolds. The notion of totally contact umbilical submanifolds was first defined by Kon [11].

A submanifold M is said to be totally contact umbilical if its second fundamental form h of M satisfies [2]

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}H + \eta(X)h(Y,\xi) + \eta(Y)h(X,\xi), \tag{3.21}$$

for any $X,\ Y \in \Gamma(TM)$, where H is a normal vector field on M (that is $H = \lambda N$, λ is a smooth function on $\mathcal{U} \subset M$). The totally contact umbilical condition (3.21) can be rewritten as,

$$h(X,Y) = B(X,Y)N = \{B_1(X,Y) + B_2(X,Y)\}N,$$

where $B_1(X,Y) = \lambda\{g(X,Y) - \eta(X)\eta(Y)\}$ and $B_2(X,Y) = \eta(X)B(Y,\xi) + \eta(Y)B(X,\xi)$.

If the $\lambda = 0$ (that is $B_1 = 0$), then the lightlike hypersurface M is said to be *totally contact geodesic* and if $B_2 = 0$, M is said to be *η-totally umbilical*.

It is easy to check that a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold is η-totally umbilical.

In the sequel, we need the following lemma.

Lemma 3.5. *Let (M,g) be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M},\overline{g})$ with $\xi \in TM$. Then, $\forall X,\ Y,\ Z \in \Gamma(TM)$,*

$$\begin{aligned}(\nabla_X B)(Y,Z) = \lambda\{B(X,Y)\theta(Z) + B(X,Z)\theta(Y)\} - \lambda\,\{\eta(Z)\overline{g}(\overline{\phi}X,\overline{\phi}Y)\\ + \eta(Y)\overline{g}(\overline{\phi}X,\overline{\phi}Z)\} + \{g(Y,Z) - \eta(Y)\eta(Z)\}X(\lambda).\end{aligned} \tag{3.22}$$

Proof. The proof follows from a direct computing using the identities (2.12) and (3.8). □

Theorem 3.6. *Let M be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$. Then $c = -1$ and λ satisfies* *the partial differential equations*

$$E(\lambda) + \lambda\tau(E) - \lambda^2 = 0, \tag{3.23}$$

$$\xi(\lambda) + \lambda(\tau(\xi) + 1) = 0, \tag{3.24}$$

$$\text{and} \quad PX(\lambda) + \lambda\tau(PX) = 0, \quad PX \neq \xi, \ \ \forall X \in \Gamma(TM). \tag{3.25}$$

Proof. Let M be a totally contact umbilical lightlike hypersurface. The direct calculation of the right hand side in (2.14) shows that, for any $X, Y \in \Gamma(TM)$,

$$\begin{aligned}(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) &= \overline{g}(\overline{R}(X,Y)Z, E)\\ &= \frac{c+1}{4}\left\{\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y)\right.\\ &\quad \left.- 2\overline{g}(\overline{\phi}X,Y)u(Z)\right\} + \tau(Y)B(X,Z) - \tau(X)B(Y,Z).\end{aligned} \tag{3.26}$$

Using (3.22), the Eq. (3.26) becomes

$$\begin{aligned}&\lambda\{B(X,Y)\theta(Z) + B(X,Z)\theta(Y)\} - \lambda\eta(Z)\{g(X,Y) - \eta(X)\eta(Y)\}\\ &\quad - \lambda\eta(Y)\{g(X,Z) - \eta(X)\eta(Z)\} + \{g(Y,Z) - \eta(Y)\eta(Z)\}X(\lambda)\\ &\quad - \lambda\{B(X,Y)\theta(Z) + B(Y,Z)\theta(X)\} + \lambda\eta(Z)\{g(X,Y) - \eta(X)\eta(Y)\}\\ &\quad + \lambda\eta(X)\{g(Y,Z) - \eta(Y)\eta(Z)\} - \{g(X,Z) - \eta(X)\eta(Z)\}Y(\lambda)\\ &= \frac{c+1}{4}\{\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}Y,X)u(Z)\}\\ &\quad + \tau(Y)B(X,Z) - \tau(X)B(Y,Z).\end{aligned} \tag{3.27}$$

Regrouping like terms in (3.27) and using (3.6), we deduce

$$\begin{aligned}&\lambda\{B(X,Z)\theta(Y) - B(Y,Z)\theta(X)\} + \lambda\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\}\\ &\quad + \{g(Y,Z) - \eta(Y)\eta(Z)\}X(\lambda) - \{g(X,Z) - \eta(X)\eta(Z)\}Y(\lambda)\\ &= \frac{c+1}{4}\{\overline{g}(\overline{\phi}Y,Z)u(X) - \overline{g}(\overline{\phi}X,Z)u(Y) - 2\overline{g}(\overline{\phi}Y,X)u(Z)\}\\ &\quad + \tau(Y)B(X,Z) - \tau(X)B(Y,Z).\end{aligned} \tag{3.28}$$

Putting $X = E$ in (3.28), we find

$$-\lambda B(Y,Z) + \{g(Y,Z) - \eta(Y)\eta(Z)\}E(\lambda) = -\frac{c+1}{4}u(Z)u(Y) - \tau(E)B(Y,Z). \tag{3.29}$$

Take $Y = Z = U$ in (3.29) we obtain $c = -1$. On the other hand, by taking $Y = V$ and $Z = U$ in (3.29), we have $(B(V,U) = \lambda)$

$$E(\lambda) + \lambda\tau(E) - \lambda^2 = 0. \tag{3.30}$$

Finally, substituting $X = PX$, $Y = PY$ and $Z = PZ$ into (3.28) with $c = -1$ and taking into account that $S(TM)$ is nondegenerate, we obtain

$$\begin{aligned}&\{PX(\lambda) + \lambda\tau(PX)\}(PY - \eta(PY)\xi) + \lambda\eta(PX)PY\\ &\quad = \{PY(\lambda) + \lambda\tau(PY)\}(PX - \eta(PX)\xi) + \lambda\eta(PY)PX.\end{aligned} \tag{3.31}$$

Putting $PX = \xi$ in (3.31), we have $\{\xi(\lambda) + \lambda(\tau(\xi) + 1)\}(PY - \eta(PY)\xi) = 0$ which leads, by taking $Y = V$, to $\xi(\lambda) + \lambda(\tau(\xi) + 1) = 0$. and the Eq. (3.24) is proved.

If PX, PY, $PZ \in \Gamma(S(TM) - \langle\xi\rangle)$, then (3.31) becomes

$$\{PX(\lambda) + \lambda\tau(PX)\} PY = \{PY(\lambda) + \lambda\tau(PY)\} PX. \tag{3.32}$$

Now suppose that there exists a vector field X_0 on some neighborhood of M such that $PX_0(\lambda) + \lambda\tau(PX_0) \neq 0$ at some point p in the neighborhood. Then, from (3.32) it follows that all vectors of the fibre $(S(TM) - \langle\xi\rangle)_p := (\overline{\phi}(TM^{\perp}) \oplus \overline{\phi}(N(TM)) \perp D_0)_p \subset S(TM)_p$ are collinear with $(PX_0)_p$. This contradicts $\dim(S(TM) - \langle\xi\rangle)_p > 1$. This implies (3.25). □

Corollary 3.7. *There exist no totally contact umbilical lightlike hypersurfaces M of an indefinite Kenmotsu manifold $(\overline{M}^c, c \neq -1)$ with $\xi \in TM$.*

A part of Theorem 3.6 is similar to that on the generic submanifold of indefinite Sasakian manifolds case given in [13]. From the Eqs. (3.23)–(3.25), the geometry of the mean curvature vector H of M is discussed. Some equations are similar to those of the indefinite Kählerian case (see [7] for details).

From (3.23)–(3.25), we have $\nabla^{\perp}_{E} H = \overline{g}(H, E)^2 N$, $\nabla^{\perp}_{\xi} H = -\overline{g}(H, E)N$ and $\nabla^{\perp}_{PX} H = 0$, $PX \neq \xi$, $\forall X \in \Gamma(TM)$. This mean that the mean curvature vector H is not parallel on M.

Note that (3.23), (3.24) and (3.25) hold when the ambient manifold $\overline{M}^c$ is replaced by an indefinite Kenmotsu space form $\overline{M}(c)$ of constant curvature $c = -1$.

Lemma 3.8. *Let M be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu space form $\overline{M}(c)$ with $\xi \in TM$. Then, the mean curvature vector H of M is $(S(TM) - \langle\xi\rangle)$-parallel, that is,*

$$\nabla^{\perp}_{PX} H = 0, \quad PX \neq \xi, \quad \forall X \in \Gamma(TM). \tag{3.33}$$

Now, we investigate the effect of semi-parallel condition on the geometry of lightlike hypersurfaces in an indefinite Kenmotsu manifold.

A submanifold M is said to be semi-parallel [5] if its second fundamental form h satisfies, for any $X, Y, X_1, X_2 \in \Gamma(TM)$,

$$(R(X, Y) \cdot h)(X_1, X_2) = -h(R(X, Y)X_1, X_2) - h(X_1, R(X, Y)X_2) = 0. \tag{3.34}$$

Theorem 3.9. *Let M be a semi-parallel lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$. Then either $c = -1$ or M is $(\overline{\phi}(TM^{\perp}), D \oplus D')$-mixed totally geodesic. Moreover, if $c = -1$, then either M is totally geodesic or $C(E, A^{*}_{E} PX) = 0$, for any $X \in \Gamma(TM)$.*

Proof. Using (2.3), (2.8), (2.14) into (3.34) and after calculation, we obtain

$$\begin{aligned}
&\frac{c-3}{4}\{g(Y, X_1)B(X, X_2) - g(X, X_1)B(Y, X_2)\}\\
&\quad + \frac{c+1}{4}\{\eta(X)\eta(X_1)B(Y, X_2) - \eta(Y)\eta(X_1)B(X, X_2) + \overline{g}(\overline{\phi}Y, X_1)B(\phi X, X_2)\\
&\quad - \overline{g}(\overline{\phi}X, X_1)B(\phi Y, X_2) - 2\overline{g}(\overline{\phi}X, Y)B(\phi X_1, X_2)\} - B(X, X_1)B(A_N Y, X_2)
\end{aligned}$$

$$+ B(Y,X_1)B(A_NX,X_2) + \frac{c-3}{4}\{g(Y,X_2)B(X,X_1) - g(X,X_2)B(Y,X_1)\}$$
$$+ \frac{c+1}{4}\{\eta(X)\eta(X_2)B(Y,X_1) - \eta(Y)\eta(X_2)B(X,X_1) + \overline{g}(\overline{\phi}Y,X_2)B(\phi X,X_1)$$
$$- \overline{g}(\overline{\phi}X,X_2)B(\phi Y,X_1) - 2\overline{g}(\overline{\phi}X,Y)B(\phi X_2,X_1)\} - B(X,X_2)B(A_NY,X_1)$$
$$+ B(Y,X_2)B(A_NX,X_1) = 0. \tag{3.35}$$

Then, by taking $X = E$ into (3.35), with the aid of $B(E,\cdot) = 0$, we have

$$\frac{c+1}{4}\{\overline{g}(\overline{\phi}Y,X_1)B(\phi E,X_2) + u(X_1)B(\phi Y,X_2) + 2u(Y)B(\phi X_1,X_2)\}$$
$$+ B(Y,X_1)B(A_NE,X_2) + \frac{c+1}{4}\{\overline{g}(\overline{\phi}Y,X_2)B(\phi E,X_1) + u(X_2)B(\phi Y,X_1)$$
$$+ 2u(Y)B(\phi X_2,X_1)\} + B(Y,X_2)B(A_NE,X_1) = 0. \tag{3.36}$$

Again, by taking $X_2 = E$ into (3.36), we get $\frac{3}{4}(c+1)u(Y)B(V,X_1) = 0$. Putting $Y = U$ into this equation, we derive $\frac{3}{4}(c+1)B(V,X_1) = 0$. Now, if $B(V,X_1) \neq 0$, $\forall X_1 \in \Gamma(D \oplus D')$, then $c = -1$. If $c \neq -1$, then $B(V,X_1) = 0$, $\forall X_1 \in \Gamma(D \oplus D')$, that is, M is $(\overline{\phi}(TM^\perp), D \oplus D')$-mixed totally geodesic.

On the other hand, suppose that $c = -1$. From (3.36) and taking $X_1 = X_2$, we obtain, $B(Y,X_1)B(A_NE,X_1) = 0$. If $B(Y,X_1) = 0$, $\forall Y, X_1 \in \Gamma(TM)$, then M is totally geodesic. If $B(Y,X_1) \neq 0$, then $B(A_NE,X_1) = 0$, that is, $C(E, A^*_E PX_1) = 0$, for any $X_1 \in \Gamma(TM)$. Thus we complete the proof. □

If M is a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$, we have

$$\overline{g}((R(E,Y)\cdot h)(X_1,E),E) = \frac{3}{4}(c+1)\lambda u(Y)u(X_1). \tag{3.37}$$

If the second fundamental form h of lightlike hypersurface M satisfies (3.34), then, we have $0 = \overline{g}((R(E,Y)\cdot h)(X_1,E),E) = \frac{3}{4}(c+1)\lambda u(Y)u(X_1)$ which leads, by taking $Y = X_1 = U$, to $\frac{3}{4}(c+1)\lambda = 0$. Therefore we have

Theorem 3.10. *Let (M,g) be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$. If the second fundamental form h of M satisfies (3.34), then $c = -1$ or M is totally geodesic.*

Using the Duggal–Bejancu Theorem 2.2, we have the following result.

Lemma 3.11. *There are no lightlike hypersurfaces of an indefinite Kenmotsu manifold $(\overline{M}^c, c \neq \frac{1}{3})$ with $\xi \in TM$ and parallel screen distribution.*

Proof. Suppose $c \neq \frac{1}{3}$ and screen distribution is parallel. From (2.3), we obtain

$$\overline{g}(\overline{R}(E,\overline{\phi}N)\overline{\phi}E,N) = \frac{3c-1}{4}. \tag{3.38}$$

On the other hand, using (2.14)

$$\overline{g}(\overline{R}(X,Y)PZ,N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)$$
$$+ \tau(Y)C(X,PZ) - \tau(X)C(YPZ). \tag{3.39}$$

From Theorem 2.2 and (3.39), we have $\overline{g}(\overline{R}(E, \overline{\phi}N)\overline{\phi}E, N) = 0$. Using this equality together with (3.38), we obtain $c = \frac{1}{3}$ which is a contradiction. □

As an example to this Lemma 3.11, we have

Example 3. Let M be a hypersurface of $\overline{M} = \{(x_1, x_2, \dots, x_7) \in \mathbb{R}^7 : x_7 \neq 0\}$, of Example 2, given by

$$x_5 = \sqrt{2}\,(x_2 + x_3)\,,$$

where $(x_1, \dots, x_7)$ is a local coordinate system for $\mathbb{R}^7$. As explained in Example 2, M is a lightlike hypersurface of $\overline{M}$ having a local quasi-orthogonal field of frames $\{U_1 = e_1, U_2 = e_2 - e_3, U_3 = E = \frac{1}{\sqrt{2}}(e_2 + e_3) - e_5, U_4 = e_4, U_5 = e_6, U_6 = \xi, N = \frac{1}{2}\{\frac{1}{\sqrt{2}}(e_2 + e_3) + e_5\}\}$ along M. Denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$. Then, we obtain

$$\overline{\nabla}_{U_3} N = -\xi, \quad \text{and} \quad \overline{\nabla}_X N = 0, \quad \forall X \in \Gamma(TM), \ X \neq U_3.$$

Using these equations above, the differential 1-form τ vanishes i.e. $\tau(X) = 0$, for any $X \in \Gamma(TM)$. So, from the Gauss and Weingarten formulae we have

$$A_N U_3 = \xi, \quad A_N X = 0, \quad \forall X \in \Gamma(TM), \ X \neq U_3, \tag{3.40}$$

$$A_E^* X = 0, \quad \nabla_X E = 0, \quad \forall X \in \Gamma(TM). \tag{3.41}$$

From (3.40) and (3.41), $C(U_3, \xi) = 1$ and $trA_E^* = 0$, i.e. the shape operator A_E^* is trace-free. Therefore, the hypersurface M of $\overline{M}$ is totally geodesic and its screen distribution is not parallel. Using the relation (3.17), $\overline{M}$, endowed with the structure $(\overline{\phi}, \xi, \eta, \overline{g})$ defined in Example 1, is of constant curvature $c = -1$.

Theorem 3.12. *Let $(M, g, S(TM))$ be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}^c$ with $\xi \in TM$, such that $S(TM)$ is parallel. If the second fundamental form h of M satisfies (3.34), then M is totally geodesic.*

Proof. Let M be a totally contact umbilical lightlike hypersurface of an indefinite Kenmotsu manifold $\overline{M}$ of constant curvature c, with $\xi \in TM$, such that $S(TM)$ is parallel. If the second fundamental form h of M satisfies (3.34), then, from (3.37), we have

$$0 = \frac{3}{4}(c+1)\lambda u(Y)u(X_1). \tag{3.42}$$

Since $S(TM)$ is parallel, from proof of the Lemma 3.11, $c = \frac{1}{3} \neq -1$. So, by putting $Y = X_1 = U$ in (3.42), we obtain $\lambda = 0$, that is M is totally geodesic and the proof is complete. □

Note that when the submanifold M is totally geodesic, that is, the second fundamental form h of M vanishes identically on M, the Eqs. (3.14) and (3.34) are trivially satisfied. This means the totally geodesic submanifold M is parallel and semi-parallel. So, we have the following result.

Theorem 3.13. *In totally contact umbilical lightlike hypersurfaces of indefinite Kenmotsu manifolds $\overline{M}^c$, tangent to the structure vector field ξ such that its screen distribution is parallel, the conditions (3.14) and (3.34) are equivalent.*

Proof. The proof follows from Theorems 3.3 and 3.12. □

It is well known that the second fundamental form and the shape operator of a non-degenerate hypersurface (in general, submanifold) are related by means of the metric tensor field. Contrary to this, we see from (2.6)–(2.11) that in the case of lightlike hypersurfaces, there are interrelations between these geometric objects and those of its screen distributions. So, the geometry of lightlike hypersurfaces depends on the vector bundles ($S(TM)$, $S(TM^\perp)$ and $N(TM)$). However, it is important to investigate the relationship between some geometrical objects induced, studied above, with the change of the screen distributions. In this case, it is known that the local second fundamental form of M on $\mathcal{U}$ is independent of the choice of the above vector bundles. This means that all results of this paper which depend only on B are stable with respect to any change of those vector bundles.

Next, we study the effect of the change of the screen distribution on the results which also depend on other geometric objects. Recall the following four local transformation equations (see [7, p. 87]) of a change in $S(TM)$ to another screen $S(TM)'$:

$$W_i' = \sum_{j=1}^{2n-1} W_i^j (W_j - \epsilon_j c_j E), \tag{3.43}$$

$$N' = N - \frac{1}{2}\left\{\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2\right\} E + \sum_{i=1}^{2n-1} c_i W_i, \tag{3.44}$$

$$\tau'(X) = \tau(X) + B(X, N' - N), \tag{3.45}$$

$$\nabla_X' Y = \nabla_X Y + B(X,Y)\left\{\frac{1}{2}\left(\sum_{i=1}^{2n-1} \epsilon_i (c_i)^2\right) E - \sum_{i=1}^{2n-1} c_i W_i\right\}, \tag{3.46}$$

where $\{W_i\}$ and $\{W_i'\}$ are the local orthonormal bases of $S(TM)$ and $S(TM)'$ with respective transversal sections N and N' for the same null section E. Here c_i and W_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, \ldots, \epsilon_{2n-1}\}$ is the signature of the basis $\{W_1, \ldots, W_{2n-1}\}$.

Denote by ω is the dual 1-form of $W = \sum_{i=1}^{2n-1} c_i W_i$ (characteristic vector field of the screen change) with respect to the induced metric g of M, that is $\omega(\cdot) = g(\cdot, W)$.

Let P and P' be projections of TM on $S(TM)$ and $S(TM)'$, respectively with respect to the orthogonal decomposition of TM. Using (3.44), it is easy to check that $C'(X, P'Y) = C'(X, PY)$, for any $X, Y \in \Gamma(TM)$.

The relationship between the second fundamental forms C and C' of the screen distributions $S(TM)$ and $S(TM)'$, respectively, is given by [using (3.44) and (3.46)]

$$C'(X, PY) = C(X, PY) - \frac{1}{2}\omega(\nabla_X PY + B(X, Y)W). \quad (3.47)$$

Note that if the lightlike hypersurface M is totally geodesic, by (3.46), the linear connection ∇ is unique.

Theorem 3.14. *Let $(M, g, S(TM))$ be a lightlike hypersurface of an indefinite Kenmotsu manifold $(\overline{M}, \overline{g})$ with $\xi \in TM$. The covariant derivatives ∇ of $h = B \otimes N$ and ∇' of $h' = B \otimes N'$ in the screen distributions $S(TM)$ and $S(TM)'$, respectively, are related as follows: for any X, Y, $Z \in \Gamma(TM)$, $\overline{g}((\nabla'_X h')(Y, Z), E) = \overline{g}((\nabla_X h)(Y, Z), E) + \mathcal{L}(X, Y, Z)$, where $\mathcal{L}$ is given by $\mathcal{L}(X, Y, Z) = B(X, Y)B(Z, W) + B(X, Z)B(Y, W) + B(Y, Z)B(X, W)$.*

We note that $\mathcal{L}(X, Y, Z)$ is symmetric with respect to X, Y and Z. Moreover $\mathcal{L}(\cdot, \cdot, E) = 0$ and $\mathcal{L}_\xi(X, Y) = \mathcal{L}(X, Y, \xi) = -u(W)B(X, Y) - u(X)B(Y, W) - u(Y)B(X, W)$. Also, it is easy to check that the parallelism of h is independent of the screen distribution $S(TM)$ ($\nabla' h' \equiv \nabla h$) if and only if the second fundamental form B of M vanishes identically on M.

Acknowledgement

The author expresses his thanks to the referee for very valuable suggestions.

References

[1] Asperti, A.C., Mercuri, F.: *Semi-parallel immersions in space forms.* Boll. Un. Mat. Ital. B (7) 8, 883–895 (1994)

[2] Bejancu, A.: *Umbilical semi-invariant submanifolds of a Sasakian manifold.* Tensor N. S. 37, 203–213 (1982)

[3] Blair, D.E.: *Riemannian Geometry of Contact and Symplectic Manifolds.* Progress in Mathematics 203, Birkhauser Boston, Inc., Boston (2002)

[4] Bonome, A., Castro, R., García-Rio, E., Hervella, L.: *Curvature of indefinite almost contact manifolds.* J. Geom. 58, 66–86 (1997)

[5] Deprez, J.: *Semi-parallel surfaces in Euclidean space.* J. Geom. 25, 192–200 (1985)

[6] Deprez, J.: *Semi-parallel hypersurfaces.* Rend. Sem. Mat. Univ. Politec. Torino 44, 303–316 (1986)

[7] Duggal, K.L., Bejancu, A.: *Lightlike Submanifolds of Semi-Riemannian Manifolds and Its Applications.* Mathematics and Applications, vol. 364. Kluwer, Dordrecht (1996)

[8] Ferus, D.: *Immersions with parallel second fundamental form.* Math. Z. 140, 87–93 (1974)

[9] Janssens, D., Vanhecke, L.: *Almost contact structures and curvature tensors.* Kodai Math. J. 4, 1–27 (1981)

[10] Kenmotsu, K.: *A class of almost contact Riemannian manifolds.* Tohoku Math. J. 24, 93–103 (1972)

[11] Kon, M.: *Remarks on anti-invariant submanifold of a Sasakian manifold.* Tensor, N. S. 30, 239–246 (1976)

[12] Kupeli, D.N.: *Singular semi-invariant Geometry*. Mathematics and Applications, vol. 366. Kluwer, Dordrecht (1996)

[13] Massamba, F.: *Totally contact umbilical lightlike hypersurfaces of indefinite Sasakian manifolds*. Kodai Math. J. 31, 338–358 (2008)

[14] Szabo, Z.I.: *Structure theorem on Riemannian spaces satisfying* $R(X,Y).R = 0$*, I: the local version*. J. Differ. Geom. 17, 531–582 (1982)

[15] Szabo, Z.I.: *Structure theorem on Riemannian spaces satisfying* $R(X,Y).R = 0$*, II: the global version*. Geom. Dedicata 19, 65–108 (1985)

Fortuné Massamba
Department of Mathematics
University of Botswana
Private Bag 0022
Gaborone
Botswana
e-mail: massfort@yahoo.fr; massambaf@mopipi.ub.bw

Received: 15 May 2008.
Revised: 30 November 2009.