

Lagrangian approach to a generalized coupled Lane–Emden system: Symmetries and first integrals

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A B S T R A C T

This paper aims to classify a generalized coupled Lane–Emden system and to compute the Noether operators corresponding to a Lagrangian for a generalized coupled Lane–Emden system which occurs in the modelling of several physical phenomena such as pattern formation, population evolution and chemical reactions. In addition the first integrals for the Lane–Emden system are constructed with respect to Noether operators.

Keywords:

Lagrangian

Noether operators

First integrals

Lane–Emden system, Gauge function

1. Introduction

In recent times, the Noether's theorem [1–4] is of paramount importance in mathematical physics. This theorem connects the symmetries and conservation laws. In fact, it provides the formula for construction of the conserved quantities (first integrals) for Euler–Lagrange differential equations once their symmetries are known. First integrals are of great value for explaining physics of the system [5] and for reducing the order of the differential equations.

Consider the generalized Lane–Emden equation

$$\frac{d^2\theta}{dx^2} + \frac{n}{x} \frac{d\theta}{dx} + \varphi(\theta) = 0, \quad (1)$$

where n is a real constant and $\varphi(\theta)$ is a real-valued continuous function of the variable θ . Eq. (1) models many problems arising in mathematical physics and astrophysics. For fixed values of n and $\varphi(\theta)$ it specifically models the theory of stellar structure, the thermal behaviour of a spherical cloud of gas, isothermal gaseous sphere and the theory of thermionic currents [6–8]. Several techniques including numerical, perturbation, Adomian's decomposition, homotopy analysis, power series and variational iteration are employed in obtaining the solution of Eq. (1) see, for example [9–11] and several studies therein.

The Noether symmetries of Eq. (1) were investigated in [12] and exact solutions for various cases which admitted Noether point symmetries were obtained. Some other works on symmetries and solutions of Lane–Emden-type equations can be

found in [13–19] and references therein. For the theory and applications of Lie group methods to differential equations, the interested reader is referred to [20–24].

The modelling of several physical phenomena such as: pattern formation; population evolution; chemical reactions; and so on (see, for example [25]), gives rise to the systems of Lane–Emden equations, and have attracted much attention in recent years. Several authors have proved existence and uniqueness results for the Lane–Emden systems [26,27] and other related systems (see, for example [28–30] and references therein).

The objective of the present study is to classify the Noether operators and to construct first integrals for the following generalized coupled Lane–Emden system

$$\frac{d^2u}{dt^2} + \frac{n}{t} \frac{du}{dt} + f(v) = 0, \quad (2)$$

$$\frac{d^2v}{dt^2} + \frac{n}{t} \frac{dv}{dt} + g(u) = 0, \quad (3)$$

where n is real constant and $f(v)$ and $g(u)$ are arbitrary functions of v and u , which is a natural extension of the well-known Lane–Emden equation.

The organization of the paper is as follows. In Section 2, we briefly recall the preliminaries of the Noether symmetry approaches. Section 3 comprises Noether operators and the associated first integrals for the system (2) and (3). Section 4 ends up with the concluding remarks.

2. Preliminaries on Noether operators and first integrals

In this section, we present some salient features of Noether operators concerning the system of two second-order ordinary differential equations, which we utilize in Sections 3. For details the reader is referred to [31–34]. Consider the vector field

$$X = \tau(t, u, v) \frac{\partial}{\partial t} + \xi(t, u, v) \frac{\partial}{\partial u} + \eta(t, u, v) \frac{\partial}{\partial v}, \quad (4)$$

which has first extension

$$X^{[1]} = X + (\dot{\xi} - \dot{u}\dot{\tau}) \frac{\partial}{\partial \dot{u}} + (\dot{\eta} - \dot{v}\dot{\tau}) \frac{\partial}{\partial \dot{v}}, \quad (5)$$

where $\dot{\tau}$, $\dot{\xi}$ and $\dot{\eta}$ denote total time derivatives of τ , ξ and η , respectively.

Let us consider the second-order system of differential equations

$$\ddot{u} = S_1(t, u, v, \dot{u}, \dot{v}), \quad \ddot{v} = S_2(t, u, v, \dot{u}, \dot{v}), \quad (6)$$

which has a Lagrangian $L(t, u, v, \dot{u}, \dot{v})$, i.e., Eq. (6) are equivalent to the Euler–Lagrange equations [34]

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}} \right) - \frac{\partial L}{\partial u} = 0, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{v}} \right) - \frac{\partial L}{\partial v} = 0. \quad (7)$$

Definition 1. The vector field X of the form (4) is called a Noether point symmetry generator corresponding to a Lagrangian $L(t, u, v, \dot{u}, \dot{v})$ of Eq. (6) if there exists a gauge function $B(t, u, v)$ such that

$$X^{[1]}(L) + D(\tau)L = D(B). \quad (8)$$

Here D is the total differentiation operator defined by [31]

$$D = \frac{\partial}{\partial t} + \dot{u} \frac{\partial}{\partial u} + \dot{v} \frac{\partial}{\partial v} + \ddot{u} \frac{\partial}{\partial \dot{u}} + \ddot{v} \frac{\partial}{\partial \dot{v}} + \dots \quad (9)$$

We now state the following theorem.

Theorem 1. Noether [1] *If X as given in (4) is a Noether point symmetry generator corresponding to a Lagrangian $L(t, u, v, \dot{u}, \dot{v})$ of Eq. (6), then*

$$I = \tau L + (\xi - \tau \dot{u}) \frac{\partial L}{\partial \dot{u}} + (\eta - \tau \dot{v}) \frac{\partial L}{\partial \dot{v}} - B \quad (10)$$

is a Noether first integral of Eq. (6) associated with the operator X .

Proof. See, e.g., [20,33]. \square

3. Noether symmetries and the associated first integrals of the system (2) and (3)

Consider the generalized coupled Lane–Emden system (2) and (3), viz.,

$$\frac{d^2 u}{dt^2} + \frac{n}{t} \frac{du}{dt} + f(v) = 0, \quad \frac{d^2 v}{dt^2} + \frac{n}{t} \frac{dv}{dt} + g(u) = 0.$$

It can readily be verified that the natural Lagrangian of system (2) and (3) is

$$L = t^n \dot{u} \dot{v} - t^n \int f(v) dv - t^n \int g(u) du. \quad (11)$$

The insertion of L from (11) into Eq. (8) and separation with respect to powers of \dot{u} and \dot{v} yields linear overdetermined system of eight PDEs. These are

$$\tau_u = 0, \quad (12)$$

$$\tau_v = 0, \quad (13)$$

$$n\tau t^{-1} + \xi_u + \eta_v - \tau_t = 0, \quad (14)$$

$$\eta_u = 0, \quad (15)$$

$$\xi_v = 0, \quad (16)$$

$$t^n \xi_t = B_v, \quad (17)$$

$$t^n \eta_t = B_u, \quad (18)$$

$$-nt^{n-1} \tau \int f(v) dv - nt^{n-1} \tau \int g(u) du - t^n \xi g(u) - t^n \eta f(v) - \tau_t t^n \int f(v) dv - \tau_t t^n \int g(u) du = B_t. \quad (19)$$

After some albeit tedious and lengthy calculations, the above system gives

$$\tau = p(t), \quad (20)$$

$$\xi = \dot{p}u - nt^{-1}pu - q_v u + a(t), \quad (21)$$

$$\eta = q(t, v), \quad (22)$$

$$B = \frac{1}{2} \dot{p} t^n u v - \frac{n}{2} \dot{p} t^{n-1} u v + \frac{n}{2} p t^{n-2} u v + t^n u \dot{k} + t^n v \dot{h} + h(t), \quad (23)$$

$$\begin{aligned} & -nt^{n-1} p \int f(v) dv - nt^{n-1} p \int g(u) du - t^n g(\dot{p}u - nt^{-1}pu - q_v u + a) - t^n f q - \dot{p} t^n \int f(v) dv - \dot{p} t^n \int g(u) du \\ & = \frac{1}{2} (\dot{p}) \cdot t^n u v - \frac{1}{2} n^2 \dot{p} t^{n-2} u v + \dot{p} n t^{n-2} u v + \frac{1}{2} n(n-2) p t^{n-3} u v + nt^{n-1} k u + t^n \dot{k} u + \dot{p} t^n v + nt^{n-1} \dot{p} v + \dot{h}. \end{aligned} \quad (24)$$

The analysis of Eq. (23) prompts the following cases:

Case 1. $n \neq 0$, $f(u)$ and $g(v)$ arbitrary but not of the form contained in cases 3, 4, 5 and 6. We find that $\tau = 0$, $\xi = 0$, $\eta = 0$, $B = \text{constant}$ and we conclude that there is no Noether point symmetry. Noether point symmetries exist in the following cases.

Case 2. $n = 0$, $f(u)$ and $g(v)$ arbitrary. We obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = 0$. Therefore we have a single Noether symmetry generator $X = \partial/\partial t$. The use of the Noether integral (10) gives the first integral

$$I = \dot{u} \dot{v} + \int f(v) dv + \int g(u) du.$$

Case 3. $f(v)$ and $g(u)$ constants. It can be easily seen that in this case we have eight Noether point symmetries associated with the standard Lagrangian for the corresponding system (2) and (3).

Case 4. $f = \alpha v + \beta$, $g = \gamma u + \lambda$, where α , β , γ and λ are constants, with $\alpha \neq 0$ and $\gamma \neq 0$.

There are three subcases, namely

4.1. For all values of $n \neq 0, 2$. We obtain $\tau = 0$, $\xi = a(t)$, $\eta = l(t)$ and $B = t^n \dot{l}u + t^n \dot{a}v - \lambda \int t^n a(t) dt - \beta \int t^n l(t) dt$. Therefore the Noether point symmetry operators are given by

$$X_1 = a(t) \frac{\partial}{\partial u} + l(t) \frac{\partial}{\partial v}, \quad (25)$$

where $a(t)$ and $l(t)$ satisfy the system $\ddot{l} + \frac{n}{t} \dot{l} + \gamma a = 0$, $\ddot{a} + \frac{n}{t} \dot{a} + \alpha l = 0$. The invocation of Theorem 1, due to Noether, gives the first integral

$$I = t^n \dot{l}u + t^n \dot{a}v - \lambda \int t^n a dt - \beta \int t^n l dt - a t^n \dot{v} - l t^n \dot{u}.$$

4.2. $n = 2$. In this subcase we obtain two Noether symmetries; X_1 given by the operator (25) with $B = t^2\dot{u} + t^2\dot{a}v$ and X_2 given by

$$X_2 = \frac{\partial}{\partial t} - ut^{-1} \frac{\partial}{\partial u} - vt^{-1} \frac{\partial}{\partial v}, \quad (26)$$

with $B = uv$.

Invoking Theorem 1, the associated first integral for X_1 is

$$I = t^2\dot{u} + t^2\dot{a}v - t^2a\dot{v} - t^2\dot{u}.$$

and the first integral corresponding to X_2 is

$$I = uv + \frac{\alpha}{2}t^2v^2 + \frac{\gamma}{2}t^2u^2 + ut\dot{v} + vt\dot{u} + t^2\dot{u}\dot{v}.$$

4.3. $n = 0$. Here we obtain five Noether operators, namely

$$X_1 = \frac{\partial}{\partial t}, \quad B = 0, \quad (27)$$

$$X_2 = \exp[-(\alpha\gamma)^{t/4}] \frac{\partial}{\partial u} - \left(\frac{\gamma}{\alpha}\right)^{1/2} \exp[-(\alpha\gamma)^{t/4}] \frac{\partial}{\partial v}, \quad (28)$$

$$B = \left(\frac{\gamma^3}{\alpha}\right)^{1/4} \exp[-(\alpha\gamma)^{t/4}]u - (\gamma\alpha)^{1/4} \exp[-(\alpha\gamma)^{t/4}]v \\ + \lambda(\gamma\alpha)^{-1/4} \exp[-(\alpha\gamma)^{t/4}] - \beta\left(\frac{\gamma}{\alpha^3}\right)^{1/4} \exp[-(\alpha\gamma)^{t/4}], \quad (29)$$

$$X_3 = \exp(\alpha\gamma)^{t/4} \frac{\partial}{\partial u} - \left(\frac{\gamma}{\alpha}\right)^{1/2} \exp(\alpha\gamma)^{t/4} \frac{\partial}{\partial v}, \quad (30)$$

$$B = \left(\frac{\gamma^3}{\alpha}\right)^{1/4} \exp(\alpha\gamma)^{t/4}u + (\gamma\alpha)^{1/4} \exp(\alpha\gamma)^{t/4}v \\ - \lambda(\gamma\alpha)^{-1/4} \exp(\alpha\gamma)^{t/4} + \beta\left(\frac{\gamma}{\alpha^3}\right)^{1/4} \exp(\alpha\gamma)^{t/4}, \quad (31)$$

$$X_4 = \cos(\gamma\alpha)^{1/4}t \frac{\partial}{\partial u} + \left(\frac{\gamma}{\alpha}\right)^{1/2} \cos(\gamma\alpha)^{1/4}t \frac{\partial}{\partial v}, \quad (32)$$

$$B = -\left(\frac{\gamma^3}{\alpha}\right)^{1/4} \sin(\gamma\alpha)^{1/4}tu - (\gamma\alpha)^{1/4} \sin(\gamma\alpha)^{1/4}tv \\ - \lambda(\gamma\alpha)^{-1/4} \sin(\gamma\alpha)^{1/4}t - \beta\left(\frac{\gamma}{\alpha^3}\right)^{1/4} \sin(\gamma\alpha)^{1/4}t, \quad (33)$$

$$X_5 = \sin(\gamma\alpha)^{1/4}t \frac{\partial}{\partial u} + \left(\frac{\gamma}{\alpha}\right)^{1/2} \sin(\gamma\alpha)^{1/4}t \frac{\partial}{\partial v}, \quad (34)$$

$$B = \left(\frac{\gamma^3}{\alpha}\right)^{1/4} \cos(\gamma\alpha)^{1/4}tu + (\gamma\alpha)^{1/4} \cos(\gamma\alpha)^{1/4}tv \\ + \lambda(\gamma\alpha)^{-1/4} \cos(\gamma\alpha)^{1/4}t + \beta\left(\frac{\gamma}{\alpha^3}\right)^{1/4} \cos(\gamma\alpha)^{1/4}t. \quad (35)$$

The corresponding first integrals are

$$I_1 = \dot{u}\dot{v} + \frac{\alpha}{2}v^2 + \frac{\gamma}{2}u^2 + \beta v + \lambda u, \quad (36)$$

$$I_2 = \left(\frac{\gamma^3}{\alpha}\right)^{1/4} \exp[-(\alpha\gamma)^{t/4}]u - (\gamma\alpha)^{1/4} \exp[-(\alpha\gamma)^{t/4}]v \\ + \lambda(\gamma\alpha)^{-1/4} \exp[-(\alpha\gamma)^{t/4}] - \beta\left(\frac{\gamma}{\alpha^3}\right)^{1/4} \exp[-(\alpha\gamma)^{t/4}] \\ - \exp[-(\alpha\gamma)^{t/4}]\dot{v} + \left(\frac{\gamma}{\alpha}\right)^{1/2} \exp[-(\alpha\gamma)^{t/4}]\dot{u}, \quad (37)$$

$$I_3 = -\left(\frac{\gamma^3}{\alpha}\right)^{1/4} \exp(\alpha\gamma)^{t/4}u + (\gamma\alpha)^{1/4} \exp(\alpha\gamma)^{t/4}v \\ - \lambda(\gamma\alpha)^{-1/4} \exp(\alpha\gamma)^{t/4} + \beta\left(\frac{\gamma}{\alpha^3}\right)^{1/4} \exp(\alpha\gamma)^{t/4} - \exp(\alpha\gamma)^{t/4}\dot{v} \\ + \left(\frac{\gamma}{\alpha}\right)^{1/2} \exp(\alpha\gamma)^{t/4}\dot{u}, \quad (38)$$

$$\begin{aligned}
I_4 = & - \left(\frac{\gamma^3}{\alpha} \right)^{1/4} \sin(\gamma\alpha)^{1/4} tu - (\gamma\alpha)^{1/4} \sin(\gamma\alpha)^{1/4} tv \\
& - \lambda(\gamma\alpha)^{-1/4} \sin(\gamma\alpha)^{1/4} t - \beta \left(\frac{\gamma}{\alpha^2} \right)^{1/4} \sin(\gamma\alpha)^{1/4} t - \cos(\gamma\alpha)^{1/4} t \dot{v} \\
& - \left(\frac{\gamma}{\alpha} \right)^{1/2} \cos(\gamma\alpha)^{1/4} t \dot{u}, \tag{39}
\end{aligned}$$

$$\begin{aligned}
I_5 = & \left(\frac{\gamma^3}{\alpha} \right)^{1/4} \cos(\gamma\alpha)^{1/4} tu + (\gamma\alpha)^{1/4} \cos(\gamma\alpha)^{1/4} tv \\
& + \lambda(\gamma\alpha)^{-1/4} \cos(\gamma\alpha)^{1/4} t + \beta \left(\frac{\gamma}{\alpha^2} \right)^{1/4} \cos(\gamma\alpha)^{1/4} t - \sin(\gamma\alpha)^{1/4} t \dot{v} \\
& - \left(\frac{\gamma}{\alpha} \right)^{1/2} \sin(\gamma\alpha)^{1/4} t \dot{u}. \tag{40}
\end{aligned}$$

Case 5. $f = \alpha v^r$, $g = \beta u^m$, where α, β are non-zero constants. There are three subcases, viz.,

5.1. $n = \frac{2m+2r+qr+3}{rm-1}$, $rm \neq 1$, $m \neq -1$, $m \neq 1$ and $r \neq -1$. We obtain $\tau = t$, $\xi = -\frac{(1+n)}{m+1}u$, $\eta = -\frac{(1+n)}{r+1}v$ and $B = \text{constant}$. Thus we obtain a single Noether point symmetry

$$X_1 = t \frac{\partial}{\partial t} - \frac{(1+n)}{m+1} u \frac{\partial}{\partial u} - \frac{(1+n)}{r+1} v \frac{\partial}{\partial v}. \tag{41}$$

Using Theorem 1, due to Noether, we obtain the associated first integral

$$I = \beta t^{n+1} \frac{u^{m+1}}{m+1} + \alpha t^{n+1} \frac{v^{r+1}}{r+1} + \frac{(n+1)}{m+1} t^n u \dot{v} + \frac{(n+1)}{r+1} t^n v \dot{u} + t^{n+1} \dot{u} \dot{v}.$$

5.2. $n = 0$, $m = -1$, $r = -1$ $\alpha = \beta$. This case provides us with two Noether symmetries, namely,

$$X_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \quad \text{and} \quad X_2 = \frac{\partial}{\partial t} \tag{42}$$

with $B = 0$ for both cases. Employing Theorem 1, we obtain the Noether's first integrals corresponding to X_1 and X_2 as

$$I_1 = \dot{u}v - u\dot{v} \quad \text{and} \quad I_2 = \dot{u}\dot{v} + \ln u + \ln v,$$

respectively.

5.3. $n = -1$, $m = -1$, $r = -1$ $\alpha = \beta$. Here we obtain two Noether symmetry operators, viz.,

$$X_1 = u \frac{\partial}{\partial u} - v \frac{\partial}{\partial v} \quad \text{and} \quad X_2 = t \frac{\partial}{\partial t} + 2u \frac{\partial}{\partial u} \tag{43}$$

with $B = 0$ and $B = -2 \ln t$, respectively. Invoking Theorem 1 we obtain the first integrals associated with X_1 and X_2 as

$$I_1 = \dot{u}vt^{-1} - u\dot{v}t^{-1} \quad \text{and} \quad I_2 = -2 \ln t + \ln u + \ln v - 2u\dot{v}t^{-1} + \dot{u}\dot{v},$$

respectively.

Case 6. $f = \alpha \exp(\beta v) + \lambda$, $g = \delta \exp(\gamma u) + \sigma$, $\alpha, \beta, \lambda, \gamma, \delta$ and σ are constants, with $\alpha \neq 0$, $\beta \neq 0$, $\delta \neq 0$, $\gamma \neq 0$. There are two subcases. They are

6.1. $n = 1$, $\lambda = 0$ and $\sigma = 0$. We obtain $\tau = t$, $\xi = -\frac{2}{\gamma}$, $\eta = -\frac{2}{\beta}$ and $B = 0$. Therefore we have a single Noether point symmetry

$$X_1 = t \frac{\partial}{\partial t} - \frac{2}{\gamma} \frac{\partial}{\partial u} - \frac{2}{\beta} \frac{\partial}{\partial v}. \tag{44}$$

The application of Theorem 1, due to Noether, results in the first integral

$$I = t^2 \dot{u}\dot{v} + \frac{\alpha t^2}{\beta} \exp(\beta v) + \frac{\delta t^2}{\gamma} \exp(\gamma u) + \frac{2}{\gamma} t \dot{v} + \frac{2}{\beta} t \dot{u}.$$

6.2. $n = 0$, $\lambda = 0$ and $\sigma = 0$. We deduce that $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = 0$. This reduces to Case 2.

Case 7. $f = \alpha \ln v + \beta$, $g = \gamma \ln u + \lambda$, where α, β, γ and λ are constants with $\alpha \neq 0$, $\gamma \neq 0$. If $n = 0$, we obtain $\tau = 1$, $\xi = 0$, $\eta = 0$ and $B = 0$. This reduces to Case 2.

4. Concluding remarks

We have studied Noether operators with respect to the standard Lagrangian of the generalized coupled Lane–Emden system (2) and (3). We obtained seven cases out of which six cases resulted in Noether point symmetries. Five new cases were

obtained and these correspond to Cases 2 and 4–7 of Section 3. The first integrals corresponding to the Noether operators in each case were constructed for the Lane–Emden system under consideration.

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