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CAPITAL BUDGETING UNDER UNCERTAINTY: AN OPTION TO INVEST

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DECLARATION

I declare that this dissertation is guided work. All sources that have been used or quoted are indicated and acknowledged by means of complete references. It is submitted in partial fulfillment of the requirements for the award of the degree of Masters of Science (Mathematics) at the University of Botswana, Gaborone, Botswana.

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Signed on : . . . . . . December 2019.

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ABSTRACT

Uncertainty and irreversibility of capital expenditures are major concerns in capital budgeting. In a highly uncertain world were management has the flexibility to adjust their operating strategy, option value is created, this option value is not captured by standard methods of capital budgeting. In this paper we review standard methods of capital budgeting such as discounted cash-flow (DCF), return on investment (ROI), payback period. Complex methods such as Monte-Carlo simulation and scenario analysis are discussed. We then look at decision tree analysis (DTA) and how it addresses the issue of flexibility and it's main shortcoming i.e. 'the discount rate problem'. Real options analysis (ROA) is discussed and we look at how it addresses the issue of managerial flexibility and how it overcomes the discount rate problem inherent in DTA. ROA is then applied to value an investment opportunity of a basalt quarry by Mbebane Enterprises. The project according to NPV analysis and IRR method found it to be a 'go' project. Using ROA, the CRR model in particular, we found option value to be significantly larger than the NPV. The binomial lattice was used as a guiding tool for timing the investment.

Chapter 1

Introduction

1.1 Background

Capital budgeting is the process of allocating resources among investment projects over a long time horizon [47]. The capital budgeting process is divided into the following stages [12]:

S1 Investment screening and selection;

S2 Capital budget proposal;

S3 Budgeting approval and authorization;

S4 Project tracking;

S5 Post-completion audit.

Since resources are limited, management are faced with the arduous task of identifying and selecting projects which are expected to generate benefits for the firm or maximize shareholder wealth [12]. Therefore the core of capital budgeting is valuing proposed projects, which is performed in stage 1 and stage 2. In stage 1, projects which are consistent with the firm's corporate strategy are selected and screened by estimating how the projects will affect future cash-flows of the firm [12]. The second stage involves creating a budget by estimating cost and revenues of the project [12]. In a non-stochastic world, evaluating proposed projects is simple, determine future cash-flows and discount these cash-flows at the risk-free rate. The problem becomes slightly more complicated if we introduce uncertainty in the project inputs, this requires more sophisticated methods in order to determine project value. According to Webster's dictionary, uncertainty or risk, is the possibility of suffering harm or loss, this is difficult to quantify. To further complicate our problem, one must also account for managerial flexibility and determine its impact on project value. This managerial flexibility refers to management's ability to mitigate risk and to take advantage of fortuitous economic circumstances. Most capital budgeting methods neglect the impact of managerial flexibility on project value. Neglecting flexibility means investment projects are grossly undervalued [29]. Another important aspect of investment projects is the issue of irreversibility (sunk costs) and how this affects the timing of investment opportunities [10]. Irreversibility, uncertainty and managerial flexibility are key issues that must be addressed when appraising investment projects especially in mining firms. In the mining industry the value of a mine/quarry is derived from the mineral resource it extracts. Commodity/mineral resource prices are very volatile and management has the capacity to alter operating strategy depending on market prices. This includes waiting to invest, contract/expand operations, shutdown/restart mining operations or abandon for salvage value. The purpose of this dissertation is to analyse an investment project by a quarry which has to deal with the issues of irreversibilty, uncertainty, having the capacity to alter its operating strategy contingent on market conditions. We first review the various capital budgeting techniques and how these methodologies fair in capturing the value of projects/assets in an uncertain economic firmament where managements have the flexibility to make mid-course corrections. We discuss discounted cash-flow (DCF) methods, return on investment, payback period, sensitivity analysis, scenario analysis, Monte-Carlo simulation, decision tree analysis. Contingent claim analysis is reviewed, in particular we look at the Black-Scholes-Merton model and the Cox, Ross and Rubinstein binomial model. We then look at real options analysis which is the application of contingent claims analysis and how this method addresses the key issues of irreversible investment, uncertainty and managerial flexibility. We then analyse an investment opportunity by a quarry using the real options methodology.

1.2 Problem Statement

Project value is dynamic in nature. This change in value is due to unpredictable, volatile market forces and managerial flexibility. According to traditional capital budgeting methods, the only source of value is cash flows. These cash flows are seen as being static across time and hence projects are treated as being analogous to bonds. This view on the source(s) of project value is narrow [27] and [33]. This particular view on project value ignores the impact of managerial flexibility on project value, consequently undervaluing investment projects [31]. This effectively reduces the quality of investment decisions.

1.3 Objectives

- To review traditional capital budgeting methods and see how they fair in capturing value inherent in investment projects with high levels of uncertainty and a dynamic, flexible management.
- To review contingent claims analysis and discuss how this can be applied to value projects.
- To build a dynamic, yet simple model which is able to capture both sources of project value and aid in timing investment projects.
- To appraise the value of a basalt mining project by Mbebane Enterprises.

Chapter 2

Traditional Capital Budgeting Methods

2.1 Discounted Cash-flow Measures (DCF)

The DCF methodology is the most widely used capital budgeting tool in project valuation. The groundwork for the methodology was laid by Irving Fischer in his two books [8]. The DCF technique is predicated on the idea that there exists a traded twin security or portfolio of securities that span the project being evaluated. This means that the value of the project under consideration can be calculated by determining the value of the twin traded security or portfolio of securities. The two main DCF methods are:

- Net Present Value (NPV) Method;
- Internal Rate of Return (IRR) Method.

We study these methods in the following sections.

2.1.1 Net Present Value (NPV)

In this DCF method the value of an asset is the present value (PV) of its expected cash flows. The present value of the investment cost is then subtracted from the present value of its cash flows to obtain the net present value (NPV). There are two ways to carry out a net present valuation, these are the risk adjusted discount rate (RADR) method and the certainty equivalent cash flows method. In the RADR method, cash flows are discounted at a market risk adjusted rate that reflects the riskiness of the project and also factors in the time value of money whilst the implementation costs are discounted at the risk-free rate. The cash flows and implementation costs are discounted at different rates because costs are subject to private (operational) risk, which is not compensated by the market, while cash flows are subject to market risk factors such as market demand, price, e.t.c [30]. However, with the certainty equivalent cash flows method, the risk adjustment is performed on the cash flows which are then discounted at the risk-free rate to account for the time value of money. For details refer to [47]. In this chapter we will focus on the RADR method which is the method of choice. The formulae for this method are given below:

$$PV = \sum_{t=1}^{N} \frac{\mathbb{E}[C_t]}{(1 + RADR)^t} \tag{2.1}$$

$$NPV = \sum_{t=1}^{N} \frac{\mathbb{E}[C_t]}{(1 + RADR)^t} - \sum_{t=1}^{N} \frac{Implementation\ Cost_t}{(1 + r)^t} \tag{2.2}$$

where:
$\mathbb{E}[C_t]$ is the expected cash flow at time t;
$RADR$ is the discount rate;
N is the project lifetime;
r is the risk-free rate.

Due to the difficulty of estimating project cash flows over long horizons, analysts normally truncate their cash flow forecast to about five years [9]. A *terminal value* is then calculated to determine the project's cash flows for the remainder of the project life. There are several methods for determining the terminal value, the method applied is based on the growth assumptions of the project's cash flows. These methods include Gordon Constant Growth Model (GCGM), Zero growth perpetuity consul, and the super-normal growth models [29]. Below are the terminal value formulas for each of the models listed.

Gordon Constant Growth Model

Cash flows are assumed to grow at a constant rate through perpetuity. The formula is given below:

$$\sum_{t=1}^{\infty} \frac{\mathbb{E}[C_{t-1}](1 + G_t)}{(1 + RADR)^t} = \frac{\mathbb{E}[C_{T-1}](1 + G_T)}{RADR - G_T} = \frac{\mathbb{E}[C_T]}{RADR - G_T}.$$

Zero Growth

Cash flows are assumed to stagnate, i.e no growth in cash flows after t. We give the formula below:

$$\sum_{t=1}^{\infty} \frac{\mathbb{E}[C_t]}{(1 + RADR)^t} = \frac{\mathbb{E}[C_T]}{RADR}.$$

Punctuated Super-normal Growth

Cash flows are assumed to experience points of super-normal growth, the formula is given below:

$$\sum_{t=1}^{T} \frac{\mathbb{E}[C_t]}{(1 + RADR)^t} + \frac{\left[\frac{\mathbb{E}[C_T](1+G_S)}{[RADR-G_S]}\right]}{(1 + RADR)^S}$$

where in the above:
t is the individual time periods;
$\mathbb{E}[C_t]$ is the free cash flow at time t;
S is the time when a punctuated growth occurs;

G is the expected growth of cash flow;
T is the terminal time for which a forecast is available;

$RADR$ is the risk adjusted discount rate.

The total asset value is the given by

$$\begin{aligned} \text{NPV} &= \text{Present Value of Forecasted Cash Flows} + \text{Terminal Value} \\ &- \text{Present Value Of Implementation Cost} \end{aligned}$$

Decision Rule
If NPV > 0 then the investment or project is worth undertaking.
If NPV $= 0$ this suggests that the project is just managing to cover the cost of capital.
If NPV < 0 this means that the investment is a poor one and should not be undertaken.

2.1.2 DCF Inputs

The DCF method discussed above is simple to compute given all the input values, but estimating these inputs is a challenge. In this subsection we take a look at the DCF inputs, their assumptions and how they are obtained.

Free Cash Flows

The cash flows of a project refer to the money going in and out as a consequence of its undertaking. This is by far the most important and most difficult input to estimate in DCF analysis. Cash flows are estimated from the projected income statement based on management perceptions about future costs, revenue, expenses, competition, and all other market conditions pertinent to the project [29]. As such these values are simply the best guess estimates based on currently available information and perceptions of management [27]. Cash flows are different from accounting income, this is because of accrual accounting. Nonetheless cash flows can be extracted from the income statement very easily using the following formula:

$$\begin{aligned} \mathbb{E}[C] &= \text{Net Income} + \text{Non-Cash Expenses} \\ &= \text{Net Income} + \text{Depreciation} + \text{Amortization} \end{aligned}$$

Due to the stochastic nature of the business firmament, estimating this input is very difficult hence the values obtained are expected values rather than an actual values. This means cash flows could be higher or lower and the degree of variation is dependent upon the volatility of these cash flows. However as [27] said

> "DCF calculations do not call for accurate forecasts, but accurate assessments of the mean of possible outcomes."

Methods used to obtain these forecasts include regression, time series models, e.t.c.

***Risk Adjusted Discount Rate* (RADR)**

This figure represents the time value of money and risk associated with a project. The most commonly used discount rate is the weighted average cost of capital (WACC). It is this discount rate that we will use throughout this dissertation. The cost of capital is the cost of financing an organization's projects, which is normally done through some optimal mix of both debt and equity [7]. Hence,

$$WACC = w_d k_d (1 - tax) + w_{ce} k_{ce} + w_{ps} k_{ps} \tag{2.3}$$

where w represents the weights of common equity ce, preferred stock ps, debt d (i.e. the various sources of capital), and k represents capital.

In order to compute a project's WACC, we must first determine the cost of equity, cost of debt, cost of preferred stock. The cost of equity is particularly difficult to quantify owing to the models used to compute it like the capital asset pricing model (CAPM), arbitrage pricing models (APM) and multi-factor models (MFM). We will focus on the CAPM as it is the most widely used. We give the formula below:

$$\begin{aligned} \text{Cost of Equity} &= \text{Risk Free Rate} + \text{Beta*Expected Risk Premium} \\ &= R_f + \beta * (\mathbb{E}[R_m] - R_f) \end{aligned}$$

***Beta* (β)**

The beta in the above formula is the sensitivity factor. It measures the impact of market fluctuations (i.e. interest rate changes, inflation, exchange rate risk, e.t.c) on the value of the project. In the CAPM, a single beta value is calculated, whilst the MFM and APM require the calculation of several of these β's (i.e measures of sensitivity of the asset with respect to market factors like interest rates, inflation, country risk, default risk e.t.c.). The higher the β the greater the sensitivity to market fluctuations.

> "In financial assets, we can obtain beta through a calculation of the covariance between a firm's stock prices and the market portfolio, divided by the variance of the market portfolio. Beta is then a sensitivity factor measuring the co-movements of a firm's equity prices with respect to the market. The problem is that equity prices change every few minutes! Depending on the time frame used for the calculation, beta may fluctuate wildly. In addition, for non-traded physical assets, we cannot

reasonably calculate beta this way. Using a firm's tradable financial assets' beta as a proxy for the beta on a single non-traded and non-marketable project within a firm that has many other projects is ill advised [29]."

In light of this, a project specific β needs to be calculated. We outline the method for determining a project specific β.

1. Locate suitable proxy companies. The proxy companies are firms whose line of business is similar to the project being undertaken.

2. Determine the equity betas of the proxy companies, their gearings and tax rates. The equity betas include both business risk and financial risk.

3. Ungear the proxy equity betas to obtain asset/project betas. Ungearing allows us to remove the financial risk of each firm to obtain asset beta which shows us business risk or project risk.

4. Calculate an average asset beta. This is the beta we use in the cost of equity calculation.

Risk-free rate

It is the interest rate of a risk-less asset (usually a government Bond with a similar duration as the project). The coupon rate of a government bond is used if the credit spread of the bond is zero, if it is non zero then the risk-free rate is given by:

$$\text{Risk-free rate} = \text{Coupon rate} - \text{Credit Spread}$$

The credit spread for government bonds can be found from ratings agencies like Moody's, S&P and Fitch.

Project lifetime

This is a value that represents how long management expects that project to be generating cash flows. This is dependent on the intrinsic nature of the asset or project being valued e.g. in mining it depends on production rate and on the mineable reserve.

2.1.3 Internal Rate of Return (IRR)

The internal rate of return is a capital budgeting metric that measures the efficiency of an investment. The IRR is based of the NPV method. In fact the IRR is that discount rate that would make NPV equal to zero. Equivalently, it can be defined as the break-even discount ratio. IRR is given by:

$$IRR \Rightarrow \sum_{t=1}^{N} \frac{\mathbb{E}[C_t]}{(1+IRR)^t} - \sum_{t=1}^{N} \frac{Implementation\ Cost_t}{(1+rf)^t} = 0 \tag{2.4}$$

Decision Rule
If the IRR > the discount rate and NPV > 0, then invest.
If the IRR < the discount rate and NPV < 0, then do not invest.

The NPV method gives the value in absolute value terms (i.e currency) whilst the IRR gives the rate of return of the project, i.e., it measures *'the bang per buck'* of an investment opportunity. The underlying assumption in IRR method is that project cash flows are reinvested at the IRR which is impractical and unrealistic. There is also the computational issue of obtaining multiple IRR's, this is problematic because the question becomes, 'which IRR is appropriate?'

2.1.4 Modified Internal Rate Of Return (MIRR)

The MIRR is an augmentation, a 'little tweak', to the traditional IRR. It seeks to solve two key issues inherent in the IRR as discussed in 2.1.3. In the MIRR, positive cash flows are reinvested at the firm's cost of capital and initial outlays are financed at the firm's financing cost whereas in the IRR it is assumed that cash flows are reinvested at the project's IRR, which is unrealistic. The solution obtained in the MIRR computation is unique, thus solving the issue of multiple IRR's. The MIRR is simple to compute, the formula is given below:

$$MIRR = \sqrt[n]{\frac{FV(Positive\ cash\ flows, cost\ of\ capital)}{PV(Initial\ Outlays, Financing\ Cost)}} \tag{2.5}$$

where n is the number of periods; FV is the future value of positive cash flows at the cost of capital, PV is the present value of all negative cash flows at the projects financing rate. Another advantage of the MIRR is that it can be used to rank projects of unequal sizes.

Advantages of the DCF

We list the following advantages of DCF as given by [30] below:

1. Clear, consistent decision criteria for all projects.
2. Same results regardless of risk preferences of investors.
3. Quantitative, decent level of precision and economically rational.
4. Not as vulnerable to accounting conventions (depreciation, inventory valuation).
5. Factors in the time value of money and basic risk factors.
6. Relatively simple, widely taught and widely accepted.
7. Simple to explain to management.

Weakness of the DCF

Like any model, the DCF is only as good as its assumptions or how well its assumptions reflect reality. In the words of Nobel Laureate Robert Solow:

> "All theory depends on assumptions which are not quite true. That is what makes it theory. The art of successful theorizing is to make the inevitable simplifying assumptions in such a way that the final results are not very sensitive. A 'crucial' assumption is one on which the conclusions do depend sensitively, and it is important that crucial assumptions be reasonably realistic. When the results of a theory seem to flow specifically from a special crucial assumption, then if the assumption is dubious, the results are suspect."

In light of this, we need to investigate the assumptions of the DCF and how they compare with business reality. Table 2.1 below compares the DCF assumptions to business reality. This table was adopted from [30].

The DCF apparatus is predicated on the idea that real investment projects are analogous to a portfolio of risk-less bonds [33]. This is clearly demonstrated by the similarity of the NPV formula and bond valuation formula. There are two key differences between the NPV formula and the risk-free bond valuation formula, these are:

- The bond valuation formula uses coupon payments instead of expected cash-flows which are used in the NPV formula. Coupon payments unlike project cash-flows are deterministic and known.
- The discount rate used in the bond valuation formula is the risk-free rate of return whereas RADR is used in the case of risky cash-flows of the NPV formula.

In the absence of managerial flexibility, this analogy holds and the two augmentations to the risk-free bond valuation formula are appropriate. But the reality is that management might find that they have the capacity to alter their operating strategy during the life of the project [47]. In particular, if market conditions are below expectations, management might have the flexibility to contract, temporarily shutdown, or abandon the project for salvage value depending on the degree of austerity of market conditions. If however, market conditions are above expectations, management might have the flexibility to expand operations. This ability to revise operating strategy changes the overall risk structure of the project. Flexibility introduces skewness in the probability distribution of asset returns since downside risk is limited, this is illustrated in Figure 2.1. The expected return (NPV) of the project is shifted to the right and project value is expanded [44]. The extra value is flexibility value. DCF ignores this extra component of value.

Assumptions	Realities
Decisions are made now and future cash flows are fixed for the future	Uncertainty and variability in future outcomes. Not all decisions are made today as some may be deferred to the future, when uncertainty becomes resolved
Projects are "mini firms" and they interchangeable with whole firms	With the inclusion of network effects, diversification, inter-dependencies, synergy, firms are portfolios of projects and their resulting cash flows. Sometimes projects cannot be evaluated as stand-alone cash flows
Once launched, all projects are passively managed	Projects are usually actively managed through project life-cycle,including checkpoints, decision options, budget constraints e.t.c
Future free cash flow streams are all highly predictable and deterministic	It may be difficult to estimate future cash flows as they are usually risky and stochastic in nature
Project discount rate used is the opportunity cost of capital which is proportional to non-diversifiable risk	There are multiple sources of business risk with different characteristics, and some are diversifiable across time and projects.
All risks are completely accounted for in the discount rate	Firm and project risk can change over the course of a project
All the factors that could affect the outcome of the project and the value to the investors are reflected in DCF though NPV or IRR	Because of project complexity and so called externalities, it may be difficult or impossible to quantify all factors in terms of incremental cash flows. Distributed, unplanned outcomes (e.g. strategic vision and entrepreneurial activity) can be significant and strategically important
Unknown, intangible or immeasurable factors are valued at zero	Many of the important benefits are intangible assets or qualitative strategic positions

Table 2.1: Disadvantages of DCF: Assumptions Vs Realities

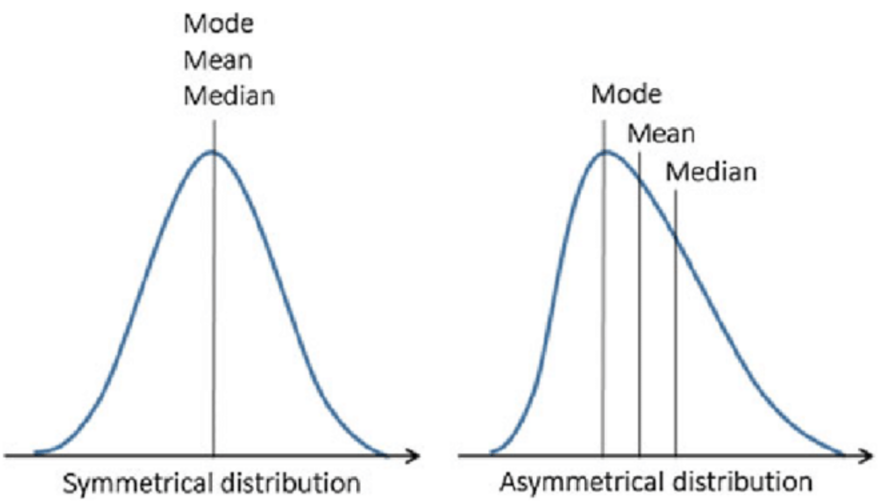


Figure 2.1: Symmetric Vs Asymmetric Distribution

Another weakness of the DCF is that it does not take into account the time series inter-dependencies of projects. These inter-dependencies introduce a compoudness in projects where one project is a necessary prerequisite to taking another project, these are called sequential/growth options. This might justify taking negative NPV projects as they might provide firms with growth options (access to new markets, technologies and business) [27]. This includes R&D projects where capital can be invested for many years without any returns or even worse negative returns but a single break though could mean super-normal returns [31].

[27] describes situations where DCF may be appropriately applied in finance and where it might not be appropriate and gives the corporate analogue. We present this in Table 2.2 below:

DCF Application in Finance	Corporate Analogue
DCF is standard for valuing bonds, preferred stocks and other fixed income securities	DCF is appropriate in valuing of safe cash-flows from contracts like financial leases.
DCF is sensible and widely used for valuing relatively safe stocks paying regular dividends	DCF is readily applied to cash cows - relatively safe businesses held for the cash they generate rather than for strategic value. It also works for capital investments like machine replacement where benefit is reduced cost in a clearly defined activity
DCF is not very useful in valuing companies with significant growth opportunities.	Capital investment projects/business that have significant opportunities in terms of growth prospects are not valued well under the DCF method
DCF method is not designed to value financial contingent claims	R&D, intangible assets are mainly option value. This option value can also be found in mining leases, the option value comes from the flexibility that the owners of the lease have to defer investment until the have observed the extent of mineral reserves as well as market conditions

Table 2.2: Application Of DCF in Finance and Corporate Analogue

In summary, when the following conditions are present in a project:

- high levels of uncertainty and
- a management that posses the flexibility to adapt and alter their operating strategy,

option value is created. This means that projects cannot be valued on a cash flow basis alone but the valuation must also capture the extra component of value brought about by managerial flexibility, i.e., the flexibility value which Stewart C. Meyers in [27] referred to as the real option value. An expanded NPV must therefore be calculated to capture both aspects of project value:

$$\text{Expanded NPV} \;=\; \text{NPV from Cash flows} \;+ \text{Real Option Value} \tag{2.6}$$

The question is, how do we capture this extra component of value? Before we endeavor to answer this question, we will look at other traditional methods of valuation and then survey the various methods that have been proposed to capture this extra component of value.

2.2 Other Traditional Methods

2.2.1 Payback Period

Another capital budgeting method is the payback period. The payback period is the number of years required to recover the initial investment outlay. If the investment consists of a series of cash outflows followed by a series of cash inflows $(C_0, C_1, C_2, \cdots, C_t)$ and $(C_{t+1}, C_{t+2}, \cdots, C_N)$, the first k for which $-\sum_{s=0}^{t} C_s \leq \sum_{r=t+1}^{k} C_r$ is the payback period [2].

This payback period is not time weighted hence it does not account for risk and time value of money. When we factor in risk and time value of money which is captured in a discount rate (e.g. WACC or some hurdle rate decided on by management) and time weight the cash flows, this is called the *Discounted Payback Period*. In this case, the payback period is the first k for which

$$-\sum_{s=0}^{t} C_s \times v_i^s \leq \sum_{r=t+1}^{k} C_r \times v_i^r$$

where v_i^s is the discount factor of project inflows which are discounted at RADR and v_i^r is the discount factor of cash outflows which are discounted at the risk-free rate.

2.2.2 Return On Investment (ROI)

Return on investment (ROI) is an accounting measure of investment return. It measures the efficiency of a project. The formula is given below

$$ROI \;=\; \frac{(Gains\ from\ Investment\ -\ Investment\ Cost)}{Investment\ Cost} \tag{2.7}$$

Unlike DCF methods ROI does not factor in the the time value of money as well as the risk associated with the project under consideration. Furthermore unlike DCF, ROI does not measure returns in terms of cash flows but accounting returns hence the value obtained is contaminated by the standard accrual accounting methodology and does not truly reflect *returns of an investment*.

2.3 Monte Carlo Simulation, Probabilistic Scenario Analysis And Decision Trees

Contrary to the assumptions of DCF and traditional capital budgeting scheme, the business environment is fraught with uncertainty and requires active management to traverse this world that is in a continual state of flux. To account for uncertainty, error in the estimation of DCF inputs as well as management flexibility, methods such as sensitivity analysis, scenario analysis, Monte-Carlo simulation as well as decision tree analysis are employed. In this section we look at these methods and how they are used as well as the shortcomings of each methodology.

2.3.1 Sensitivity Analysis

Sensitivity analysis (SA) is a typical technique used to quantify the impact of parameter uncertainty on overall simulation/prediction uncertainty (Helton, 1993; Saltelli et al., 2000) [50]. The sensitivity, S, of a parameter, P, is defined below as follows:

$$S = \frac{\partial x/x}{\partial P/P} \tag{2.8}$$

where x is the state variable under consideration. Sensitivity analysis is a powerful tool for examining issues relating to uncertainties in model structure, or in input or parameter values [23]. The inputs of a model are the proposed values of any parameter or state variable whose true value is uncertain/unknown. According to [23], the objective of sensitivity analysis is three-fold:

1. to estimate the uncertainty in the model's predictions caused by uncertainty in the values of inputs.
2. to examine the consequences of varying the model's structure on its generality and predictive power.
3. to determine the degree to which inaccuracies in their assumed values could lead to serious errors in prediction.

The first and third point are of particular interest in financial modelling specifically capital budgeting as most of the model parameters (inputs) cannot be known with certainty. Sensitivity Analysis is a tool used in financial modeling to analyze how the different values of a set of independent variables affect a specific dependent variable under certain specific conditions. It is the case in capital budgeting that

> "...the estimates of cash flows used in capital budgeting are invariably derived from (oftentimes most likely) forecasts of other primary variables (the project's life, salvage value, production costs, the price of the product, the size and growth of the market, the firms market share,etc.)" [47]

It behooves the prudent analyst to carry out a sensitivity analysis on the discounted cash flow model. Setting the cash-flows, NPV or IRR as the resulting variables, we then perturb each precedent variables or parameter and note the change in the resulting variables. The precedent variables include revenues, costs, tax rates, discount rates, capital expenditures, depreciation, and so forth (in other), which ultimately flow through the model to affect the cash-flow, net present value or IRR figure [31]. Each variable is perturbed and varied by a predefined amount and the resulting change in net revenues is captured. This approach is great for understanding which variables drive or impact the bottom line the most. The uncertain key variables that drive the net present value and, hence, the decision are called *critical success drivers.*

Tornado Diagram

The results of a sensitivity analysis can be represented graphically on a diagram known as a tornado diagram. This diagram lists the precedent variables in descending order of magnitude, i.e. from the most sensitive variables at the top to the least sensitive at the bottom. The degree of magnitude on a resultant variable is represented by the length of the bars going to the left and the right to show impact of precedent variable when each precedent variable is increased and decreased by a certain amount. The greater the impact, the longer the bars, hence the chart has the longest bars at the top and the shortest at the bottom, giving it the tornado like shape.

2.3.2 Scenario Analysis

Scenario Analysis is a capital budgeting tool used to analyse investment projects by considering multiple probable circumstances and observes how the project performs under each of these scenarios. In scenario analysis, we estimate expected cash flows and asset value under various scenarios, with the intent of getting a better sense of the effect of risk on value. Unlike sensitivity analysis where only a single input is altered and the rest held constant, scenario analysis allows the analyst to alter all the relevant inputs to align with an alternate reality. The most basic scenario analysis has two scenarios, that is, the best case scenario, and worst case scenario. According to [8] this can be carried out in two ways:

Method 1
The inputs in the DCF can all be set to their most optimistic (best case) values for the best case scenario and set to their most pessimistic (worst case) values for the worst case scenario. Performing the analysis this way may be infeasible as it does not consider inter-dependencies

between inputs e.g a firm may need to lower prices to increase revenue while lowering margins.

Method 2
Under this method inter-dependencies between variables are considered and the best case and worst case scenarios are defined in terms of what is realistic (feasible) hence we look for a combination of inputs that maximize value e.g. find a combination of growth and margin that maximizes value for the firm.

This basic type of scenario analysis allows firms to determine the riskiness of an asset by taking the difference between the best and worst case scenario adjusted to size. It also allows firms that are concerned about potential spill over effects of an investment to gauge the impact of the investment going bad on their operations by using the worst case scenario.

The basic 2 state scenario analysis can be extended to include multiple scenarios. In its most general form, the value of a risky asset can be computed under a number of different scenarios, varying the assumptions about both macroeconomic and asset-specific variables [8]. Below is an outline of how to carry out a multiple scenario analysis, this step by step outline is sourced from [8].

Step 1 Decide which factors the scenarios will be built around: This can be done though the aid of sensitivity analysis.

Step 2 Determine how many scenarios to analyze for each factor.

Step 3 Estimate asset cash flows under each scenario.

Step 4 Assign probabilities to each scenario.
These scenarios and probabilities can then be used to determine an expected NPV if the scenario's are exhaustive (i.e. if the scenario's account for all the possible combinations of inputs, or simply put the probabilities sum up to one). Creating and analyzing project value under an exhaustive list of possible outcomes is nigh impossible without computational methods line Monte-Carlo simulation which we deal with in the next section. This is especially true if input values are random variables that take values on a continuous time scale (i.e. face continuous uncertainty)

Although scenario Analysis allows us to vary multiple inputs and observe how the project behaves in different situations, it still does not account for managerial flexibility. Moreover scenario analysis is better suited for dealing with risk that takes the form of discrete outcomes than risk that takes value on a continuum. An example of the former is a shift in regulatory rules, changes in margins or market share are an example of the latter [8]. Evidently this is a major reason why it is difficult to create an exhaustive list of possible scenario's and obtain an expected NPV. Most input values do not take discrete values but take on values on a continuous scale, each with its own probability distribution, scenario analysis then falls short since we cannot see the full range of possible outcomes.

2.3.3 Monte Carlo Simulation

A simulation is an approximate imitation of the functioning of a process or system by means of the functioning of another [17]. According to Naylor et al in [36],

> "Simulation is a numerical technique for conducting experiments on a digital computer, which involves certain types of mathematical and logical models that describe the behaviour of a business or economic system (or some component thereof) over extended periods of time."

We can simulate both deterministic and probabilistic system. One method for performing these simulations is the Monte Carlo Method (also known as the method of statistical trials). According to [31], Monte Carlo is a stochastic tool that helps people think probabilistically and not deterministically (in terms of certainty). This is very important especially in the field of finance and business which is fraught with uncertainty. Probabilistic models are useful for critical situations only, that is, situations where deterministic models are inefficient or irrelevant [13]. These situation are ubiquitous in finance, the fundamental sciences and engineering. But what is Monte Carlo Simulation? According to [3]

> "Monte-Carlo simulation consists of solving various problems of computational mathematics by means of the construction of some random process for each such problem, with the parameters of the process equal to the required quantities of the problem. These quantities are then determined approximately by means of observations of the random process and the computation of its statistical characteristics, which are approximately equal to the required parameters."

This process is carried out by computers. Since computers are deterministic, how then does it generate random events (random numbers)? Since computer programs simply execute algorithms, given an initial result, we are able to guess the outcome from the onset. This is at odds with random number generation. Therefore the key problem in Monte Carlo simulation is creating a computer algorithm that is chaotic enough to produce unpredictable results, i.e. from the observers point of view [15].

> Thus, to build a dynamically chaotic algorithm is sufficient to generate random events[15].

These algorithms are called pseudo-random number generators. Pseudo random numbers are collections of numbers that are produced by a deterministic algorithm and yet seem to be random in the sense that, en masse, they have appropriate statistical properties [14]. [15] shows how uniform random variables can be used to generate random variables for which we know the distribution of. This then simplifies the problem of generating random variables on computer to generating a uniform random variable. In order to generate a uniform random variable without resorting to a complex chaotic dynamic, all we need to do is to build an algorithm such that the generated result has the statistical regularity of a uniform random variable [15]. The details of how to generate these Uniform Random variables can be found in [3], [15] and [36].

[36] list some reasons why simulation might be appropriate, we list some of them below:

- Simulation makes it possible to study and experiment with complex internal interactions of a given system whether it be a firm, an industry, an economy, or some subsystem of one of these.
- Through simulation we can study the effects of certain informational, organisational, and environmental changes on the operation of a system by making alterations in the model of the system and observing the effects of these alterations on the system's behaviour.
- Detailed observation of the system being simulated may lead to a better understanding of the system and to suggestions for improving it, suggestions that otherwise would not be apparent.
- Simulation of complex systems can yield valuable insight into which variables are more important than others in the system and how these variables interact.
- Simulation can be used to experiment with new situations about which we have little or no information so as to prepare for what may happen.
- Simulation can be used as a pedagogical device to reinforce analytical solution methodologies.

Monte Carlo Simulations have a wide range of application and are used in diverse fields; from Neuroscience to engineering, from physics to finance. [36] and [13] outlines situations where simulation is appropriate:

- It may be impossible or extremely expensive to obtain data from certain processes in the real world. This includes situations like the effect of tax cuts in the economy, the effect of an advertising campaign on total sales [36].
- The observed system may be so complex that it cannot be described in terms of a set of mathematical equations for which analytic solutions are obtainable. Most economic systems fall into this category [36].
- Simulation can be used in situations where a mathematical model can be used to model a system of interest, but it may be difficult to find solutions by straight forward analytic techniques. Economic systems fall into this category [36].

- It may be either impossible or very costly to perform validating experiments on the mathematical models describing the system. Alternative hypotheses can be tested by the simulation data [36].

- A physical experiment which can be described by well-established physical laws, but for which certain model parameters are difficult to calibrate with accuracy [13].

- A physical experiment which can be described by well-established physical laws, but which is submitted to a huge number of rapidly and unpredictably varying forces [13].

- A phenomenon driven by incompletely known physical laws [13].

- A phenomenon which is not driven by physical laws and involves a huge number of heterogeneous sources of changes [13].

Probability theory allows us to model complex phenomena whose states cannot be precisely deduced from accurate measurements. Classical methods cannot be used such phenomena. Stochastic modeling involves the choosing of probability distributions of the inputs in the system and aims to compute the probability distributions of important characteristics of the phenomena under consideration. In the simplest sense, Monte Carlo simulation creates artificial futures by generating many sample paths of outcomes (thousands into the hundreds of thousands even) and analyzes their prevalent characteristics [31]. A simulation calculates numerous scenarios of a model by repeatedly picking values from a user-predefined probability distribution for the uncertain variables and using those values for the model [31]. As all those scenarios produce associated results in a model, each scenario can have a forecast. Forecasts are events (usually with formulas or functions) that you define as important outputs of the model. In finance, particularly in the traditional capital budgeting scheme, Monte Carlo Method is used to simulate cash flows or NPV's of a capital investment project [47]. Cash flows or NPV's are set as outputs of the system, the probability distributions of the crucial primary variables are preset, along with the formula's defining the inter-dependencies and correlations between system inputs. The simulation is then run which yields a distribution of the output variables.

However, Monte Carlo simulation is not without its shortcomings. [47] lists the following:

- It may be difficult to capture all the inter-dependencies of inputs even if analysts are unbiased. Expert help may be needed to capture this complexity.

- If the simulation outcome is a distribution of NPV's, it is not clear what discount rate should be used, the meaning of this distribution becomes unclear. Moreover we can no longer think of the present value of a project to be the price a project would command under perfect capital markets.

- There is no rule for translating the probability distribution of NPV into clear cut decision.

- There exists the temptation for simulation users to use the variance of project outcomes as a measure of risk. This variance is a measure of the projects total risk.
- The extreme values of simulated outputs are unreliable
- Simulation does not factor in managerial flexibility (i.e. managements ability to review their initial operating strategy) in its calculations, it follows the predefined operating strategy built into it from the onset and blindly follows this predefined strategy.

2.3.4 Decision Tree Analysis (DTA)

The major limitation of DCF as mentioned in previous sections is its inability to factor managerial flexibility (i.e. the ability of managements to make mid course corrections), this flexibility introduces asymmetry in project returns. The asymmetry adds an extra component of value which we call the "***real option value***", this option value is not captured by standard DCF techniques. For this reason decision tree analysis (DTA) has been traditionally used to evaluate projects with uncertain cash flows when the decision maker can follow more than one decision path [49].

> "DTA helps management structure the decision problem by mapping out all feasible alternative managerial actions contingent on the possible states of nature (chance events) in a hierarchical manner. As such it is particularly useful for analyzing complex sequential investment decisions when uncertainty is resolved at distinct, discrete points in time [47]."

A decision tree is a pictorial representation of the decision paths managements can take and how they respond to different outcomes in the market.

According to [8] a decision tree is made up of multiple nodes, these are:

1. Root Node is the beginning of a decision tree where management faces an uncertain outcome or decision choice. It is the goal of the analysis to determine what the value of the an investment is at this point.
2. Decision Node are points where a decision maker chooses among possible (alternative) routes contingent on the outcomes of previous decisions.
3. Event Nodes are the possible outcomes of risky gamble.
4. End Nodes represent the final outcomes of prior investment decisions in response to chance events.

[47] mentions only two nodes (i.e. decision points) which are the decision node, i.e., the decision points under the control of management; the other is the event node (which he calls the outcome

node) which are the decision points for "*nature*" and the decision maker has no control over these outcomes. The decision maker is like a supplicant waiting for the judgment of a most unpredictable whimsical god,"nature", and makes do with what he is given. In the words of [47],

> "It is as if management is playing a "game" (say, chess) against nature, taking turns to make their "moves". (Nature, of course is the unthinking opponent making its moves randomly so such a game against nature presents management with an optimization problem,...)"

[9] outlines steps to creating a decision tree, which are given below:

1. Divide analysis into risk phases: This is a critical step in decision tree analysis where you outline all the risks you will encounter in the future.

2. In each phase, estimate the probabilities of the outcomes: these probabilities should add up to 1.

3. Define decision points: At these points managements determines best course of action based on previous decisions and their outcomes as well as expectations about future outcomes.

4. Compute cash flows/value at end nodes: cash flows are estimated at each end node and then discounted to arrive at a present value.

5. Fold back the tree: this is the final step in decision tree analysis where expected values are computed by working backwards in the tree. At an event node the expected value is a weighted probability average of the different outcomes. At a decision node however the expected value is computed for each branch and the greatest value among the various expected values is chosen, this is the optimal decision.

Consider the following simple example. You are offered a choice of accepting P65 now or partaking in a gamble in which you have a 60% chance of winning P110 and a 40% chance of winning P55 which you will receive a year later, the risk free rate is 10%. Figure 2.2 below shows the decision tree for this offered gamble.

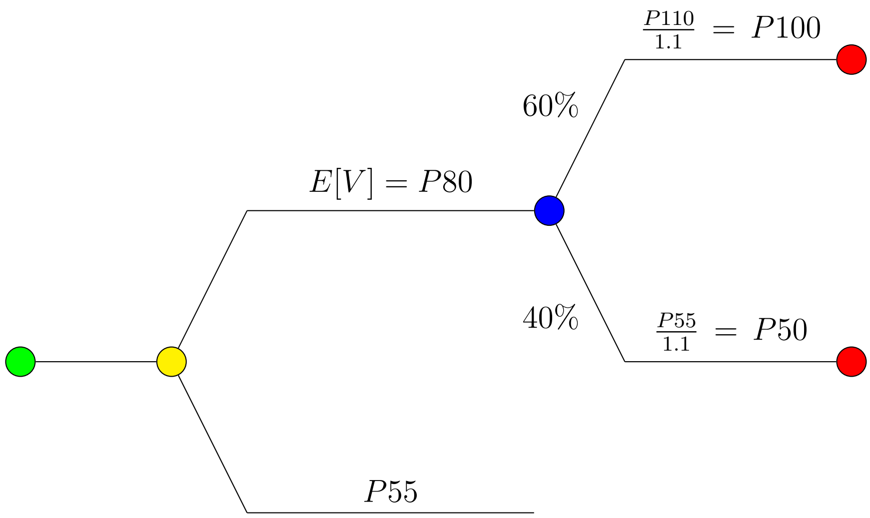


Figure 2.2: Simple Decision Tree

Where the green circle represents the root node, yellow circle represents decision node, blue circle represents event nodes and red circles represent end nodes.

DTA views chance events as though they occurred at discrete points in time whereas resolution of uncertainty might take place continuously over time. The main shortcoming of the DTA method is in determining the appropriate discount rate used in folding back through the decision tree where present values of the investment are computed [27]. Since managerial flexibility introduce asymmetry in project returns and changes the overall risk structure, prudence demands the analyst to compute a new discount rate which reflects the change in the risk profile [47]. In the example above the risk free rate is used because no risks are faced and thus we only need to account for the time value of money. This is not the case in complex, volatile capital investments which are subject to market forces like interest rate, exchange rate, demand fluctuations e.t.c. [27] in his ground breaking work was the first to point out that the tools for capturing these "options" in capital investment could already be found in contingent claims analysis (options pricing). Managerial flexibility creates an extra component of value which he called the ***real option value***. He argued that the tools of financial options pricing theory could be applied to capital investment projects to value these ***real options***. In the next sections we review options pricing theory and how it can be applied to real assets.

Chapter 3

Review Of Option Pricing Methodology

Definition 1. *An option is a security giving the right to the holder buy or sell an asset, which is called thee underlying, subject to certain conditions, within a certain period of time. These conditions include:*

C1 Expiration Date denoted (T) *refers to the last day on which the option can be exercised.*

C2 Strike Price denoted (K) *is the price that is paid for the underlying at the time of exercise.*

C3 Terms of exercise.

There are two basic types of options, these are:

- ***Put option***, $P(S, t)$, which gives the holder the right to sell the underlying asset subject to C1, C2 and C3.
- ***Call option***, $C(S, t)$, which gives its holder the right to buy the underlying asset, subject to C1, C2 and C3.

Puts and calls are collectively called plain vanilla options, this is due to the simple nature of their payoffs.

Definition 2. *The style of the option refers to the terms of exercise or exercise rules which are outlined in the option contract which along with the strike price* (K)*, and expiration date constitute the option. There are two main styles of options, these are:*

- *European Style Options: These options can only be exercised at the maturity date* T *of the option.*
- *American Style Options: These options can be exercised at any time upto and including its expiration date* T*.*

There are many more styles of options which are basically hybrids of the two main styles, these include Bermudian, canary, Verde options e.t.c.

Definition 3. *The* ***Option Premium*** *is the price paid for an option.*

Options provide limited downside risk for the holders of these contracts but unlimited upside benefit. This is why an option premium is charged. The question is how to price these unique derivatives. Options pricing theory is a framework used to determine this price.

There are several methods used to determine the price of an option, these include:

- lattice Methods e.g. Cox, Ross and Rubinstein (CRR) Binomial Model.
- Closed form solutions e.g. Black-Scholes-Merton (BSM) model
- Partial differential equations
- Variance reduction techniques
- Monte-Carlo methods

We will focus on lattice metods, specifically the CRR binomial model and the closed for, solution of the BSM. We begin with a discussion of the one-period binomial model and extend these ideas to the multi-period case, we will then look at the celebrated Black-Scholes-Merton formula. Convergence of the CRR to the BSM is then shown. We then extend the BSM and CRR to allow for dividends to conclude our discussion on options pricing.

3.1 Binomial Options Pricing Model

In this section we discuss the Binomial options pricing Model which was developed by Cox, Ross and Rubinstein in their journal article Options Pricing: A simplified Approach. We first list the assumptions underlying this model below:

A1 No market frictions.

A2 No credit risk.

A3 Competitive and well-functioning markets.

A4 No intermediate cash flows (no dividends).

A5 No arbitrage opportunities.

A6 No interest rate uncertainty i.e. interest rates r remain constant.

A7 Underlying Asset follows a binomial process which means trading occurs at discrete points in time and at each discrete point asset will either move up by some factor $U > 1$ with probability p or down by some factor $D > 0$ with probability $(1 - p)$

We begin our discussion of the single period model, which gives us intuition and sets a good foundation for the underlying economics and ideas used to price options. We then extend the one period model to the more complex multi-period model as discussed by Cox, Ross and Rubinstein.

3.1.1 Single Period Binomial Model

Under the CRR assumptions, we describe and define all the necessary objects of our model below.

Definition 4. *A market in the one-period model consists of two $(n+1)$-dimensional vectors*

$$X(0) = (X_0(0), X_1(0), \cdots, X_n(0)) \text{ and } X(T) = (X_0(T), X_1(T), \cdots, X_n(T))$$

where $X_0(t), X_1(t), \cdots, X_n(t)$ represent the prices of $n+1$ securities at time $t=0$ and $t=T$. $X_0(t)$ represents the price of a safe investment (risk-free bond) and $X_1(t), \cdots, X_n(t)$ represent the prices of n risky assets.

Definition 5. *A portfolio in the one-period model is an (n + 1)-dimensional deterministic vector*

$$\theta = (\theta_0, \theta_1, \cdots, \theta_n)$$

where θ_i represents the number of units held in security i at time $t = 0$; $i = 0, 1, \cdots, n$

Definition 6. *The value of a portfolio at time $V^\theta(t)$, is given by*

$$V^\theta(t) = \theta \cdot X(t) = \sum_{i=0}^{n} \theta_i X_i(t)$$

Definition 7. *An arbitrage opportunity is a portfolio θ such that*

1. $V^\theta(0) = 0$
2. $V^\theta(T) \geq 0$
3. $\mathbb{E}_{\mathbb{P}}[V^\theta(T)] > 0$, *where $\mathbb{P}$ is a probability measure.*

Model Specification

- Let $T = 1$
- There are three securities in our market, namely:
  - A stock (risky asset) denoted $S(n, j)$.
  - A Bond (risk free asset) $\beta(n, j)$.

– A Contingent Claim (derivative asset) which we denote as $W(n, j)$.

where n represents the nth time step, and j represents number of up moves (hence describes the state of the security); (n, j) represents the state of the security at the nth time step. For the one period model $n = 1$ and $j \in \{0, 1\}$.

- The time $t = 0$ price of the stock is is $S(0, 0)$ (which is known) and the time $t = 1$ price of the stock $S(1, j)$ which assumes one of two possible values, $S(1, 1) = US(0, 0)$ with probability p, i.e., the price of the stock in the upstate or $S(1, 0) = DS(0, 0)$ with probability $(1 - p)$, i.e., the price in a down state.

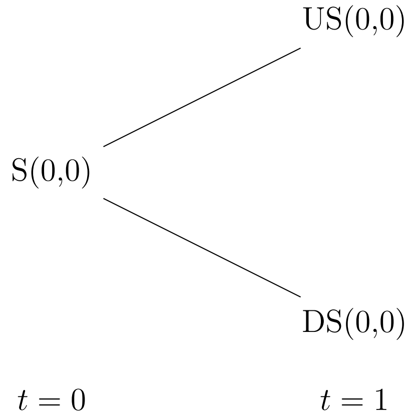


Figure 3.1: 1-Period Binomial Model

- The time $t = 0$ price of the bond is $\beta(0, 0)$ and the time $t = 1$ price is $R\beta(0, 0)$; where $R = e^{r \cdot 1}$.

- $V^\theta(n, j)$ denotes wealth of the investor in state j at the nth time step.

Theorem 3.1.1. *There is no arbitrage in our market if the following inequality is satisfied*

$$0 < D < R < U.$$

Proof. By definition we have that $U > D > 0$. Now if we let $D \geq R$ then we could sell a bond β with interest rate r and use the funds to purchase stock, in this case we are assured that even if the stock moves down we will have enough to cover our debt (i.e. payoff the bond) at $t = 1$ this implies that to prevent arbitrage $D < R$ If we let $U < R$ We can short sell the stock and invest in the bond at $t = 0$ which will always outperform the stock, we will receive $R\beta$ at $t = 1$ which we can use to buy back the stock and return it to its owner. This results in a riskless gain, thus we require that $R < U$.
Hence we have the result. □

Theorem 3.1.2. *Let CRR assumptions hold and $0 < D < R < U$. Then the time $t = 0$ price of a European Option in the One period Binomial Model is*

$$V^{\theta}(0,0) = \frac{1}{R}[\pi W(1,1) + (1-\pi)W(1,0)].$$

Proof. Beginning with an initial wealth $V^{\theta}(0,0)$, we purchase Δ_0 shares leaving us with a cash position of $V^{\theta}(0) - \Delta_0$ which is invested in bonds. Hence, our portfolio at time $t = 0$ is given by

$$\theta = \left(\frac{V^{\theta}(0,0) - \Delta_0}{\beta(0,0)}, \Delta_0\right)$$

where $\beta(0,0)$ is the price of a bond at $t = 0$ Our time $t = 1$ wealth (value of our portfolio) is

$$V^{\theta}(1,j) = R(V^{\theta}(0,0) - \Delta_0 S(0,0)) + \Delta_0 S(1,j).$$

We want to choose $V^{\theta}(0,0)$ and Δ_0 such that $W(1,1) = V^{\theta}(1,1)$ and $W(1,0) = V^{\theta}(1,0)$, replication of the contingent claim requires that

$$\begin{aligned} W(1,1) &= R(V^{\theta}(0,0) - \Delta_0 S(0,0)) + \Delta_0 S(1,1) \\ W(1,0) &= R(V^{\theta}(0,0) - \Delta_0 S(0,0)) + \Delta_0 S(1,0) \end{aligned}$$

which can be rewritten as

$$\frac{1}{R}W(1,1) = V^{\theta}(0,0) + \left(\frac{1}{R}S(1,1) - S(0,0)\right)\Delta_0; \tag{3.1}$$

$$\frac{1}{R}W(1,0) = V^{\theta}(0,0) + \left(\frac{1}{R}S(1,0) - S(0,0)\right)\Delta_0. \tag{3.2}$$

We have a system of two equations and two unknowns which we solve by elimination. Subtracting 3.2 from 3.1, we have

$$\frac{W(1,1) - W(1,0)}{R} = \left(\frac{S(1,1) - S(1,0)}{R}\right)\Delta_0$$

$$\Delta_0 = \frac{W(1,1) - W(1,0)}{S(1,1) - S(1,0)}. \tag{3.3}$$

Substituting 3.3 into 3.1 gives

$$\frac{1}{R}W(1,1) = V^{\theta}(0,0) + \left(\frac{1}{R}S(1,1) - S(0,0)\right)\frac{W(1,1) - W(1,0)}{S(1,1) - S(1,0)}.$$

Solving for $V^{\theta}(0,0)$ gives

$$V^{\theta}(0,0) = \frac{1}{R}\left[\left(\frac{RS(1,0) - S(1,0)}{S(1,1) - S(1,0)}\right)W(1,1) + \left(\frac{S(1,1) - RS(0,0)}{S(1,1) - S(0,0)}\right)W(1,0)\right].$$

Letting

$$\pi = \left(\frac{RS(1,0) - S(1,0)}{S(1,1) - S(1,0)}\right) \qquad ; \qquad 1 - \pi = \left(\frac{S(1,1) - RS(0,0)}{S(1,1) - S(0,0)}\right)$$

we have

$$V^{\theta}(0,0) = \frac{1}{R}[\pi W(1,1) + (1 - \pi)W(1,0)]. \tag{3.4}$$

□

3.1.2 Multi-period

We extend the ideas of one-period model here and provide justification for the options pricing formula of Cox, Ross and Rubinstein. We describe the necessary objects for this exposition.

Model Specification

- Given the expiration date T, we define N as the number of steps and $\delta t = \frac{T}{N}$ as the step size. There are $N + 1$ terminal nodes and 2^N possible price paths.
- The price of a stock $S(n,j)$ (risky asset) $S(n,j) = S(0,0)U^j D^{n-j}$, where n represents the number of time steps, and j represents number of up moves (hence describes the state of the security); (n,j) represents the state of the security at the nth time step; where $n \in \{0, 1, \cdots, N\}$ and $j = 0, 1, \cdots, n$.
- $R = e^{r\delta t}$.

We illustrate a 3-period binomial model in Figure 3.2 below.

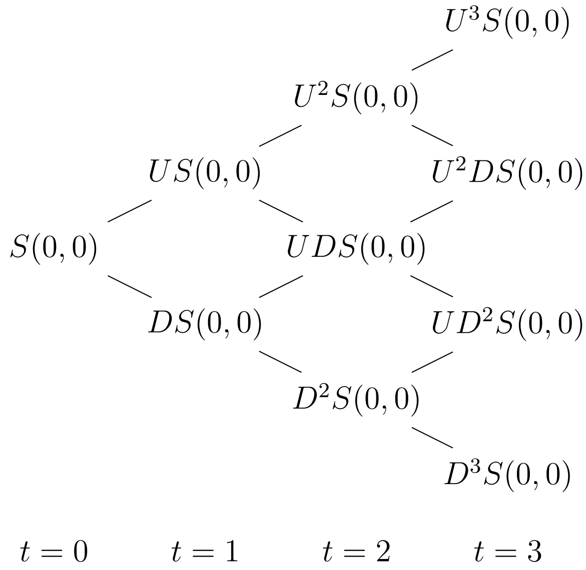


Figure 3.2: 3-Period Binomial Model

The pricing of a derivative instrument in the multi-period setting (i.e. with N time periods), can be thought of solving n one-period binomial models backwards recursively in time at each time step. We lay out the technical details below.

Theorem 3.1.3. *Let the CRR assumptions hold and $0 < D < R < U$. Then the arbitrage price of a European Option $W(n,j)$ in the binomial model is given by*

$$W(n,j) = \frac{1}{R^{N-n}} \sum_{l=0}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} W(N, j+l).$$

Proof. We shall use mathematical induction.

For $N = n+1$

$$W(n,j) = \frac{1}{R}\left[\pi W(n+1, j+1) + (1-\pi) W(n+1, j)\right] \tag{3.5}$$

which is the one period binomial formula, hence shown for $N = n+1$.

For $N = n+2$

$$W(n,j) = \frac{1}{R}\left[\pi W(n+1, j+1) + (1-\pi) W(n+1, j)\right] \tag{3.6}$$

$$W(n+1,j+1) = \frac{1}{R}\left[\pi W(n+2, j+2) + (1-\pi) W(n+2, j+1)\right] \tag{3.7}$$

$$W(n+1,j) = \frac{1}{R}\left[\pi W(n+1, j+1) + (1-\pi) W(n+1, j)\right]. \tag{3.8}$$

Substituting 3.7 and 3.8 into 3.6 we have

$$W(n,j) = \frac{1}{R^2}\left(\pi^2 W(n+2,j+2) + 2\pi(1-\pi)W(n+2,j+1) + (1-\pi)^2 W(n+1,j)\right)$$
$$W(n,j) = \frac{1}{R^2}\sum_{l=0}^{2}\binom{2}{l}\pi^l(1-\pi)^{2-l}W(N,j+l)). \tag{3.9}$$

Assume true true for $N = n+k$, i.e.

$$W(n,j) = \frac{1}{R^k}\sum_{l=0}^{k}\binom{k}{l}\pi^l(1-\pi)^{k-l}W(n+k,j+l) \tag{3.10}$$

Let $N = n+k+1$.
We have that

$$W(n+k,j+l) = \frac{1}{R}\left[\pi W(n+k+1,j+l+1) + (1-\pi)W(n+k+1,j+1)\right]. \tag{3.11}$$

Substituting 3.11 into 3.10 gives the following

$$W(n,j) = \frac{1}{R^k}\sum_{l=0}^{k}\binom{k}{l}\pi^l(1-\pi)^{k-l}\left(\frac{1}{R}\left[\pi W(n+k+1,j+l+1) + (1-\pi)W(n+k+1,j+l)\right]\right)$$

$$\begin{aligned} W(n,j) = \frac{1}{R^{k+1}}\Bigg[&\sum_{l=0}^{k}\binom{k}{l}\pi^{l+1}(1-\pi)^{k-l}W(n+k+1,j+l+1) \\ &+ \sum_{l=0}^{k}\binom{k}{l}\pi^l(1-\pi)^{k+1-l}W(n+k+1,j+l)\Bigg]. \end{aligned}$$

$$
\begin{aligned}
W(n,j) = \frac{1}{R^{k+1}} \Bigg[& \binom{k}{0}\pi^1(1-\pi)^k W(n+k+1, j+1) \\
& + \binom{k}{1}\pi^2(1-\pi)^{k-1} W(n+k+1, j+2) + \cdots \\
& + \binom{k}{k-1}\pi^k(1-\pi)^1 W(n+k+1, j+k) \\
& + \binom{k}{k}\pi^{k+1}(1-\pi)^0 W(n+k+1, j+k+1) \\
& + \binom{k}{0}\pi^0(1-\pi)^{k+1} W(n+k+1, j) \\
& + \binom{k}{1}\pi^1(1-\pi)^k W(n+k+1, j+1) + \cdots \\
& + \binom{k}{k-1}\pi^{k-1}(1-\pi)^2 W(n+k+1, j+k-1) \\
& + \binom{k}{k}\pi^k(1-\pi)^1 W(n+k+1, j+k) \Bigg].
\end{aligned}
$$

Rearranging and collecting like terms, gives

$$
\begin{aligned}
W(n,j) = \frac{1}{R^{k+1}} \Bigg[& \binom{k}{0}\pi^0(1-\pi)^{k+1} W(n+k+1, j) \\
& + \left\{ \binom{k}{0} + \binom{k}{1} \right\} \pi^1(1-\pi)^k W(n+k+1, j+1) \\
& + \left\{ \binom{k}{1} + \binom{k}{2} \right\} \pi^2(1-\pi)^{k-1} W(n+k+1, j+2) \\
& + \left\{ \binom{k}{2} + \binom{k}{3} \right\} \pi^3(1-\pi)^{k-2} W(n+k+1, j+3) + \cdots \\
& + \left\{ \binom{k}{k-2} + \binom{k}{k-1} \right\} \pi^{k-1}(1-\pi)^2 W(n+k+1, j+k-1) \\
& + \left\{ \binom{k}{k-1} + \binom{k}{k} \right\} \pi^k(1-\pi)^1 W(n+k+1, j+k) \\
& + \binom{k}{k}\pi^{k+1}(1-\pi)^0 W(n+k+1, j+k+1) \Bigg].
\end{aligned}
$$

$$\begin{aligned}
W(n,j) = \frac{1}{R^{k+1}} \Bigg[&\binom{k}{0} \pi^0 (1-\pi)^{k+1} W(n+k+1, j) \\
&+ \binom{k+1}{1} \pi^1 (1-\pi)^k W(n+k+1, j+1) \\
&+ \binom{k+1}{2} \pi^2 (1-\pi)^{k-1} W(n+k+1, j+2) \\
&+ \binom{k+1}{3} \pi^3 (1-\pi)^{k-2} W(n+k+1, j+3) + \cdots \\
&+ \binom{k+1}{k-1} \pi^{k-1} (1-\pi)^2 W(n+k+1, j+k-1) \\
&+ \binom{k+1}{k} \pi^k (1-\pi)^1 W(n+k+1, j+k) \\
&+ \frac{1}{R^{k+1}} \binom{k}{k} \pi^{k+1} (1-\pi)^0 W(n+k+1, j+k+1) \Bigg].
\end{aligned}$$

which simplifies to

$$W(n,j) = \frac{1}{R^{k+1}} \sum_{l=0}^{k+1} \binom{k+1}{l} \pi^l (1-\pi)^{k+1-l} W(n+k+1, j+l).$$

□

Lemma 3.1.1. *The arbitrage price of a European call option, $C(N,l)$, with underlying asset, S, with expiry T and strike K at the nth time step is given*

$$C(n,j) = \frac{1}{R^{N-n}} \sum_{l=0}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} C(N,l)$$

where

$$C(N,l) = [S(N,l) - K]^+ = [S(n,j) U^l D^{N-n-l} - K]^+.$$

Corollary 3.1.1. *The arbitrage price of a European call option, $C(N,l) = [S(N,l) - K]^+$, written on a stock, S, at the nth time step with expiry time T and strike K in the Binomial Model is given by the CRR option pricing formula*

$$C(n,j) = S(n,j) \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \hat{\pi}^l (1-\hat{\pi})^{N-n-l} - \frac{K}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l}$$

where

$$\hat{k} = \inf\{k \in \mathbb{N} : k > \log(K/(S(n,j)D^{N-n}))/\log(U/D)\}$$

and

$$\hat{\pi} = \frac{\pi U}{R} \in (0, 1).$$

Proof. We have that

$$\begin{aligned}
S(n,j)U^k D^{N-n-k} - K > 0 &\Leftrightarrow S(n,j)U^k D^{N-n-k} > K \\
(U/D)^k &\Leftrightarrow K/(S(n,j)D^{N-n}) \\
\log\left((U/D)^k\right) &\Leftrightarrow \log\left(K/(S(n,j)D^{N-n})\right) \\
k &\Leftrightarrow \log\left(K/(S(n,j)D^{N-n})\right)/\log(U/D)
\end{aligned}$$

if $\hat{k} > N - n$ then $C(n,j) = 0$. However, if $\hat{k} < N - n$, we have

$$\begin{aligned}
C(n,j) &= \frac{1}{R^{N-n}} \sum_{l=0}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} C(N,l) \\
C(n,j) &= \frac{1}{R^{N-n}} \sum_{l=0}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} [S(n,j)U^l D^{N-n-l} - K]^+ \\
C(n,j) &= \frac{1}{R^{N-n}} \sum_{l=0}^{\hat{k}} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} 0 \\
&+ \frac{1}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} (S(n,j)U^l D^{N-n-l} - K) \\
C(n,j) &= \frac{1}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} S(n,j)U^l D^{N-n-l} \\
&- \frac{1}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} K \\
C(n,j) &= S(n,j) \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \left[\frac{\pi U}{R}\right]^l \left[\frac{(1-\pi)D}{R}\right]^{N-n-l} \\
&- \frac{K}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l} \\
C(n,j) &= S(n,j) \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} (\hat{\pi})^l (1-\hat{\pi})^{N-n-l} \\
&- \frac{K}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l}
\end{aligned}$$

$$0 < \frac{\pi U}{R} = \left(\frac{R-D}{U-D}\right)\frac{U}{R} = \left(\frac{R-D}{U-D}\right)\frac{\frac{1}{R}}{\frac{1}{U}} = \frac{(1-\frac{D}{R})}{(1-\frac{D}{U})} < 1.$$

□

Definition 8. *We define a random variable R_i below:*

$$R_i = \begin{cases} 1, & \textit{up move with probability } p \\ 0, & \textit{down move with probability } (1-p) \end{cases} \tag{3.12}$$

R_i is a Bernoulli random variable with parameter p, this implies that:

$$\mathbb{E}[R_i] = p$$

$$Var[R_i] = p(1-p).$$

We can rewrite $S(n,j) = S(0,0)U^j D^{n-j}$ as $S(n,j) = S(0,0)U^{\sum_{i=1}^n R_i} D^{n-\sum_{i=1}^n R_i}$.

Lemma 3.1.2. *The values of the up and down parameters in the CRR model are given by*

$$u = e^{\sigma\sqrt{\delta t}} \qquad ; \qquad d = e^{-\sigma\sqrt{\delta t}}.$$

Proof.

$$\begin{aligned} S(n\delta t) &= S_0 U^{\sum_{i=1}^n R_i} D^{n-\sum_{i=1}^n R_i} && (3.13)\\ \frac{S(n\delta t)}{S_0} &= U^{\sum_{i=1}^n R_i} D^{n-\sum_{i=1}^n R_i} \\ \log\left(\frac{S(n\delta t)}{S_0}\right) &= \log\left(U^{\sum_{i=1}^n R_i} D^{n-\sum_{i=1}^n R_i}\right) && (3.14)\\ &= \log\left(U^{\sum_{i=1}^n R_i}\right) + \log\left(D^{n-\sum_{i=1}^n R_i}\right) \\ &= \left(\sum_{i=1}^n R_i\right)\log(U) + \left(n - \sum_{i=1}^n R_i\right))\log(D) \\ &= n\log(D) + \left(\sum_{i=1}^n R_i\right)\log\left(\frac{U}{D}\right). \end{aligned}$$

Let $\phi = \log\left(\frac{S(n\delta t)}{S_0}\right)$.

We thus have

$$\begin{aligned}
\mathbb{E}[\phi] &= n\log(D) + \left(\sum_{i=1}^{n} E[R_i]\right)\log\left(\frac{U}{D}\right) \\
\hat{\mu}n &= n\log(D) + \left(\sum_{i=1}^{n} p\right) log\left(\frac{U}{D}\right) \\
&= np\log(U) + n(1-p)\log(D) \qquad (3.15)
\end{aligned}$$

and

$$\begin{aligned}
Var[\phi] &= \left[\log\left(\frac{U}{D}\right)\right]^2 \sum_{i=1}^{n} Var[R_i] \\
&= \left[\log\left(\frac{U}{D}\right)\right]^2 \sum_{i=1}^{n} p(1-p) \\
\hat{\sigma}^2 n &= \left[\log\left(\frac{U}{D}\right)\right]^2 np(1-p). \qquad (3.16)
\end{aligned}$$

Letting $n \to \infty$ we have

$$np\log(U) + n(1-p)log(D) \to \mu t \qquad (3.17)$$

$$\left[\log\left(\frac{U}{D}\right)\right]^2 np(1-p) \to \sigma^2 t \qquad (3.18)$$

this implies that for large n

$$p\log(U) + (1-p)\log(D) \approx \mu\frac{t}{n} \qquad (3.19)$$

$$\log(U) - \log(D) \approx \sqrt{\frac{\sigma^2 \frac{t}{n}}{p(1-p)}}. \qquad (3.20)$$

Rearranging and letting $\delta t = \frac{t}{n}$, we have

$$p\log(U) + (1-p)\log(D) \approx \mu\delta t \qquad (3.21)$$

$$\log(U) \approx \log(D) + \sqrt{\frac{\sigma^2\delta t}{p(1-p)}}. \qquad (3.22)$$

Substituting 3.22 into 3.21 and rearranging we have

$$\log(D) \approx \mu\delta t - p\sqrt{\frac{\sigma^2\delta t}{p(1-p)}}. \tag{3.23}$$

Letting $p = (1-p) = 1/2$

$$\log(D) \approx \mu\delta t - \sigma\sqrt{\delta t}. \tag{3.24}$$

This implies

$$D \approx e^{\mu\delta t - \sigma\sqrt{\delta t}} \tag{3.25}$$

which implies

$$log(U) \approx \mu\delta t - \sigma\sqrt{\delta t} + 2\sigma\sqrt{\delta t} \tag{3.26}$$

and hence

$$U \approx e^{\mu\delta t + \sigma\sqrt{\delta t}}. \tag{3.27}$$

Letting $n \to \infty$, $\delta t \to 0$ and we have

$$U = e^{\sigma\sqrt{\delta t}} \quad ; \quad D = e^{-\sigma\sqrt{\delta t}}.$$

□

We state without proof the following theorem:

Theorem 3.1.4. *Given the price of a European call option $C(n,j)$ with strike K and expiry date T, we can determine the price of a put option $P(n,j)$ with the exact same strike and expiry date using the following formula,*

$$C(n,j) - P(n,j) - S(n,j) + KR^{N-n} = 0.$$

3.2 Continuous Time

In this section we look at the celebrated Black-Scholes-Merton formula popularly known as the Black-Scholes formula for pricing contingent claims, in particular we discuss the European Call Option. We first outline the underlying assumptions below:

A1 No market frictions.

A2 No credit risk.

A3 Competitive and well-functioning markets.

A4 No intermediate cash flows (no dividends).

A5 No arbitrage opportunities.

A6 No interest rate uncertainty i.e. interest rates r remain constant.

A7 Stock has constant Volatility.

A8 No jumps.

A9 Trading takes place continuously.

Model Specification

- There are 3 securities being traded in our market; a stock whose price is denoted S_t, a call option whose price is denoted $C(S, t)$ and a risk-free bond whose price is denoted β_t :
- The Stock price, S_t, follows a Geometric Brownian Motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{3.28}$$

  where: μ is the drift, σ is the volatility of the stock price and $B_t \sim \mathcal{N}(0, 1)$ is a Brownian Motion.

- Bond price, β_t, is given by:

$$d\beta_t = r\beta_t dt \tag{3.29}$$

  where r is the risk-free rate.

- The value of a portfolio denoted V_t is given by:

$$V_t = a_t S_t + b_t \beta_t \tag{3.30}$$

where a_t is the number of units of stock held in the portfolio; and b_t is the number of units of the bond in the portfolio. We require that this portfolio be self-financing. The self financing condition is given by

$$dV_t = a_t dS_t + b_t d\beta_t. \tag{3.31}$$

Theorem 3.2.1 (Ito's Formula)**.** *Let X_t be a Ito process given by*

$$dX_t = udt + vdB_t.$$

Let $g(t,x) \in C^2([0,\infty] \times \mathbb{R})$ i.e. (g is twice continuously differentiable on $([0,\infty] \times \mathbb{R})$. *Then*

$$Y_t = g(t,x)$$

is also an Ito process, and

$$dY_t = \frac{\partial g(t,X_t)}{\partial t}dt + \frac{\partial g(t,X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g(t,X_t)}{\partial x^2}(dX_t)^2,$$

where $(dX_t)^2 = (dX_t)\cdot(dX_t)$ is computed according to the rules the rules

$$dt \cdot dt = 0 \qquad dt \cdot dB_t = 0 \qquad dB_t \cdot dB_t = dt.$$

Theorem 3.2.2 (Black Scholes PDE)**.** *Let the BSM assumptions hold, then the dynamics of a European Call option are modeled by the following backward parabolic heat equation:*

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0.$$

Proof. This proof closely follows the one given by [41]. Given that $C(S,t)$ is the price of a European call option at time t, By Ito's Formula, we have that

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S}(dS)^2 \tag{3.32}$$

where $C = C(S,t)$ and $S = S_t$. Substituting 3.28 into 3.32, we have

$$\begin{aligned} dC &= \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}(\mu S + \sigma S dB_t) + \frac{\partial^2 C}{\partial S^2}(\mu S dt + \sigma S dB_t)^2 \\ dC &= \left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dB_t. \end{aligned}$$

From 3.31,

$$dV = a_t dS + b_t d\beta_t.$$

Substituting 3.28 into 3.31, we have the following:

$$\begin{aligned} dV &= a_t(\mu S dt + \sigma S dB_t) + b_t r\beta_t dt \\ &= (a_t\mu S + b_t r\beta_t)dt + a_t\sigma S dB_t. \end{aligned}$$

We want $V = C$, that is the portfolio to replicate the call options payoff at each time t, this implies $dV = dC$, hence

$$(a_t\mu S + b_t r\beta_t)dt + a_t\sigma S dB_t = \left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2}\right)dt + \sigma S\frac{\partial C}{\partial S}dB_t$$

By comparing coefficients, we have

$$a_t = \frac{\partial C}{\partial S} \tag{3.33}$$

$$a_t\mu S + b_t r\beta_t = \frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2}. \tag{3.34}$$

Substituting 3.33 into 3.34

$$\begin{aligned} \frac{\partial C}{\partial S}\mu S + b_t r\beta_t &= \frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} \\ b_t r\beta_t &= \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} \\ b_t &= \frac{1}{r\beta_t}\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2}\right) \end{aligned} \tag{3.35}$$

From the replication argument $V = C$, we have the following

$$C = a_t S + b_t\beta_t. \tag{3.36}$$

Substituting 3.33 and 3.35 into 3.36, we get

$$\begin{aligned} C &= \frac{\partial C}{\partial S}S + \frac{1}{r\beta_t}\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2}\right)\beta_t \\ rC &= rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} \\ 0 &= \frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC \end{aligned}$$

which is the Black-Scholes PDE.

□

Theorem 3.2.3 (Black Scholes Merton Formula)**.** *Given that the dynamics of the price of a European call option are described by the PDE*

$$\frac{1}{2}\sigma^2 S\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} + \frac{\partial C}{\partial t} - rC = 0.$$

With the following boundary conditions: $C(0,t) = 0$; $C(S,t) \sim S\, as\, S \to \infty$ *and* $C(S,t) = max(S-K,0)$ *The Solution to the Backward Parabolic PDE is,*

$$C(S,t) = S\mathcal{N}(d_1) + Ke^{-r\tau}\mathcal{N}(d_2)$$

where:

$$d_1 = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}} \qquad ; \qquad d_2 = \frac{\ln(S/K) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}$$

and

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{x^2}{2}}dx.$$

Proof. This proof follows the one given by [37]. We use a change of variable to solve this backward parabolic heat equation.

Let

$$t = T - \frac{\tau}{\sigma^2/2} \tag{3.37}$$
$$S = Ke^x \tag{3.38}$$
$$C(S,t) = K\nu(x,\tau). \tag{3.39}$$

Making τ in 3.37 and x in 3.38 subject of their respective formulas give:

$$\tau = \frac{\sigma^2}{2}(T-t) \tag{3.40}$$
$$x = \ln(\frac{S}{K}). \tag{3.41}$$

Differentiating 3.39 w.r.t S and τ, gives

$$\frac{\partial C}{\partial t} = K\frac{\partial \nu}{\partial \tau}\frac{\partial \tau}{\partial t} = K\frac{-\sigma^2}{2}\frac{\partial \nu}{\partial \tau} \tag{3.42}$$
$$\frac{\partial C}{\partial S} = K\frac{\partial \nu}{\partial x}\frac{\partial x}{\partial S} = \frac{K}{S}\frac{\partial \nu}{\partial x}. \tag{3.43}$$

We then differentiate 3.43 w.r.t S, gives

$$\begin{aligned}\frac{\partial^2 C}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right) &= \frac{\partial}{\partial S}\left(\frac{K}{S}\frac{\partial \nu}{\partial x}\right)\\ &= \frac{-K}{S^2}\frac{\partial \nu}{\partial S} + \frac{K}{S}\frac{\partial}{\partial S}\left(\frac{\partial \nu}{\partial x}\right)\\ &= \frac{-K}{S^2}\frac{\partial \nu}{\partial S} + \frac{K}{S}\frac{\partial}{\partial x}\left(\frac{\partial \nu}{\partial x}\right)\frac{\partial x}{\partial S}\\ &= \frac{-K}{S^2}\frac{\partial \nu}{\partial S} + \frac{K}{S^2}\frac{\partial^2 \nu}{\partial x^2}\end{aligned} \tag{3.44}$$

The terminal condition $C(S,T) = \max(S_T - K, 0)$ becomes $\max(Ke^x - K, 0)$ and $C(S,T) = K\nu(x,0)$ so $\nu(x,0) = \max(e^x - 1, 0)$.

Substituting 3.39, 3.42, 3.43 and 3.44 into the Black-Scholes PDE gives

$$0 = \frac{-\sigma^2}{2}\frac{\partial \nu}{\partial \tau} + \frac{\sigma^2}{2}S^2\left(\frac{-K}{S^2}\frac{\partial \nu}{\partial x} + \frac{K}{S^2}\frac{\partial^2 \nu}{\partial x^2}\right) + rS\left(\frac{K}{S}\frac{\partial \nu}{\partial x}\right) - rK\nu \tag{3.45}$$

$$\frac{\sigma^2}{2}\frac{\partial \nu}{\partial \tau} = \frac{\sigma^2}{2}\left(-\frac{\partial \nu}{\partial x} + \frac{\partial^2 \nu}{\partial x^2}\right) + r\frac{\partial \nu}{\partial x} - r\nu. \tag{3.46}$$

We then divide through by $\frac{\sigma^2}{2}$

$$\frac{\partial^2 \nu}{\partial x^2} - \frac{\partial \nu}{\partial x} + \frac{r}{\sigma^2/2}\frac{\partial \nu}{\partial x} - \frac{r}{\sigma^2/2}\nu = \frac{\partial \nu}{\partial \tau}. \tag{3.47}$$

Letting $k = \frac{r}{\sigma^2/2}$

$$\frac{\partial^2 \nu}{\partial x^2} + (k-1)\frac{\partial \nu}{\partial x} - k\nu = \frac{\partial \nu}{\partial \tau}. \tag{3.48}$$

We then let

$$\nu = e^{\alpha x + \beta \tau}u(x,\tau), \tag{3.49}$$

this gives

$$\frac{\partial \nu}{\partial \tau} = \beta e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial \tau} \tag{3.50}$$

$$\frac{\partial \nu}{\partial x} = \alpha e^{\alpha x + \beta \tau}u + e^{\alpha x + \beta \tau}\frac{\partial u}{\partial x} \tag{3.51}$$

and

$$\frac{\partial^2 \nu}{\partial x^2} = \alpha^2 e^{\alpha x+\beta\tau} u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}. \tag{3.52}$$

Substituting 3.50, 3.51 and 3.52 into 3.48, gives

$$\beta e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau} = \alpha^2 e^{\alpha x+\beta\tau} u + 2\alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2} \tag{3.53}$$

$$+ (k-1)(\alpha e^{\alpha x+\beta\tau} u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}) - k e^{\alpha x+\beta\tau} u. \tag{3.54}$$

Dividing though by $e^{\alpha x+\beta\tau}$

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)\left(\alpha u + \frac{\partial u}{\partial x}\right) - ku. \tag{3.55}$$

Collecting like terms gives

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + [2\alpha + (k-1)]\frac{\partial u}{\partial x} + [\alpha^2 + (k-1)\alpha - k - \beta]u.$$

We then choose

$$\alpha = -\frac{(k-1)}{2} \quad ; \quad \beta = -\frac{(k+1)^2}{4} \tag{3.56}$$

which reduces our equation to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{3.57}$$

which is a one dimensional heat equation.

The initial condition is transformed to

$$\begin{aligned} u(x,0) &= e^{(\frac{(k-1)}{2})x}\nu(x,0) \\ &= e^{(\frac{(k-1)}{2})x}\max(e^x - 1, 0) \\ &= \max(e^{(\frac{(k+1)}{2})x} - e^{(\frac{(k-1)}{2})x}, 0). \end{aligned}$$

We observe that $u_0(x) = u(x,0) > 0$ when $x > 0$ otherwise $u(x,0) = 0$.
From the study of PDE's we know that the solution of the one dimensional heat equation is given by

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{\infty} u_0(s)e^{-(\frac{(x-s)^2}{4\tau})}ds. \tag{3.58}$$

Now let

$$z = \frac{(s-x)}{\sqrt{2\tau}}$$

which implies

$$s = z\sqrt{2\tau} + x \qquad ; \qquad ds = \sqrt{2\tau}dz.$$

Then

$$u(s,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(z\sqrt{2\tau} + x)e^{-\frac{z^2}{2}} dz. \tag{3.59}$$

Since $u_0 = (x) = u(x,o) > 0$ when $x > 0$ otherwise $u(x,o) = 0$ we will integrate over the domain where $u_0 > 0$ i.e. $z > -\frac{x}{\sqrt{2\tau}}$, on this domain $u_0 = e^{(\frac{(k+1)}{2})(z\sqrt{2\tau}+x)} - e^{(\frac{(k-1)}{2})(\sqrt{2\tau}+x)}$. Hence integral 3.59 becomes

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} \left[e^{(\frac{(k+1)}{2})(z\sqrt{2\tau}+x)} - e^{(\frac{(k-1)}{2})(\sqrt{2\tau}+x)} \right] e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} \left[\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{(k+1)}{2})(z\sqrt{2\tau}+x)} e^{-\frac{z^2}{2}} dz - \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{(k-1)}{2})(\sqrt{2\tau}+x)} e^{-\frac{z^2}{2}} dz \right]. \end{aligned}$$

Let

$$I_1 = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{(k+1)}{2})(z\sqrt{2\tau}+x)} e^{-\frac{z^2}{2}} dz \qquad I_2 = \frac{1}{\sqrt{2\tau}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{(k-1)}{2})(\sqrt{2\tau}+x)} e^{-\frac{z^2}{2}} dz.$$

We first solve for I_1

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{(k+1)}{2})(z\sqrt{2\tau}+x)} e^{-\frac{z^2}{2}} dz \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z^2-(k+1)\sqrt{2\tau}z)-(k+1)x)} dz. \end{aligned}$$

By completing the square of the exponent

$$
\begin{aligned}
-\frac{1}{2}(z^2-(k+1)\sqrt{2\tau}z)-(k+1)x) &= -\frac{1}{2}(z^2-(k+1)\sqrt{2\tau}z)-(k+1)x \\
&+\frac{(k+1)^2\tau}{2}-\frac{(k+1)^2\tau}{2}) \\
&= \frac{1}{2}\left(z-\frac{(k+1)^2\sqrt{2}\tau}{2}\right)^2+\frac{(k+1)^2\tau}{4}+\frac{(k+1)x}{2}.
\end{aligned}
$$

I_1 becomes

$$
\begin{aligned}
I_1 &= \frac{e^{\frac{(k+1)^2\tau}{4}+\frac{(k+1)x}{2}}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}\left(z-\frac{(k+1)^2\sqrt{2}\tau}{2}\right)^2}dz \\
&= \frac{e^{\frac{(k+1)^2\tau}{4}+\frac{(k+1)x}{2}}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}\left(z-(k+1)^2\sqrt{\tau/2}\right)^2}dz.
\end{aligned}
$$

Letting

$$
y = z-(k+1)^2\sqrt{\tau/2} \qquad \Rightarrow \qquad dy = dz
$$

I_1 becomes

$$
I_1 = \frac{e^{\frac{(k+1)^2\tau}{4}} + \frac{(k+1)x}{2}}{\sqrt{2\pi}}\int_{-\frac{x}{\sqrt{2\tau}}-\sqrt{\tau/2}(k+1)}^{\infty} e^{-\frac{y^2}{2}}dy.
$$

The cumulative normal distribution is given by

$$
\begin{aligned}
\mathcal{N}(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-\frac{y^2}{2}}dy \\
&= \frac{1}{\sqrt{2\pi}}\int_{-d}^{\infty} e^{-\frac{y^2}{2}}dy.
\end{aligned}
$$

Letting $d_1 = -\frac{x}{\sqrt{2\tau}}-\sqrt{\tau/2}(k+1)$, I_1 becomes

$$
I_1 = e^{\frac{(k+1)^2\tau}{4}+\frac{(k+1)x}{2}}\mathcal{N}(d_1).
$$

Solving for I_2

$$
\begin{aligned}
I_2 &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{(\frac{(k-1)}{2})(z\sqrt{2\tau}+x)} e^{-\frac{z^2}{2}} dz \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}(z^2-(k-1)\sqrt{2\tau}z)-(k-1)x)} dz.
\end{aligned}
$$

By completing the square

$$
\begin{aligned}
-\frac{1}{2}(z^2-(k-1)\sqrt{2\tau}z)-(k-1)x) &= -\frac{1}{2}(z^2-(k-1)\sqrt{2\tau}z)-(k-1)x \\
&\quad + \frac{(k-1)^2\tau}{2} - \frac{(k-1)^2\tau}{2}) \\
&= \frac{1}{2}\left(z - \frac{(k-1)^2\sqrt{2}\tau}{2}\right)^2 + \frac{(k-1)^2\tau}{4} + \frac{(k-1)x}{2}.
\end{aligned}
$$

I_2 becomes

$$
\begin{aligned}
I_2 &= \frac{e^{\frac{(k-1)^2\tau}{4}+\frac{(k-1)x}{2}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}\left(z-\frac{(k-1)^2\sqrt{2}\tau}{2}\right)^2} dz \\
&= \frac{e^{\frac{(k-1)^2\tau}{4}+\frac{(k-1)x}{2}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}\left(z-(k-1)^2\sqrt{\tau/2}\right)^2}.
\end{aligned}
$$

We let

$$
w = z-(k-1)^2\sqrt{\tau/2} \qquad \Rightarrow \qquad dw = dz
$$

and I_2 becomes

$$
I_2 = \frac{e^{\frac{(k-1)^2\tau}{4}+\frac{(k-1)x}{2}}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}-\sqrt{\tau/2}(k-1)}^{\infty} e^{-\frac{y^2}{2}} dw.
$$

Using the idea of cumulative normal and letting $d_2 = -\frac{x}{\sqrt{2\tau}} - \sqrt{\tau/2}(k-1)$, I_2 can be rewritten as

$$
I_2 = e^{\frac{(k-1)^2\tau}{4}+\frac{(k-1)x}{2}} \mathcal{N}(d_2).
$$

This means

$$u(x,\tau) = e^{\frac{(k+1)^2\tau}{4}+\frac{(k+1)x}{2}}\mathcal{N}(d_1) - e^{\frac{(k-1)^2\tau}{4}+\frac{(k-1)x}{2}}\mathcal{N}(d_2).$$

And hence,

$$\begin{aligned}
\nu(x,\tau) &= e^{\alpha x+\beta\tau}u(x,\tau) \\
&= e^{(-\frac{(k-1)}{2})x-(\frac{(k+1)^2}{4})\tau}\left[e^{\frac{(k+1)^2\tau}{4}+\frac{(k+1)x}{2}}\mathcal{N}(d_1) - e^{\frac{(k-1)^2\tau}{4}+\frac{(k-1)x}{2}}\mathcal{N}(d_2)\right] \\
&= e^{x}\mathcal{N}(d_1) - e^{-k\tau}\mathcal{N}(d_2) \\
V(S,t) &= K\nu(x,\tau) \\
&= Ke^{\ln(\frac{S}{K})}\mathcal{N}(d_1) - Ke^{\frac{r}{\sigma^2/2}\sigma^2/2(T-t)}\mathcal{N}(d_2) \\
&= S\mathcal{N}(d_1) - Ke^{r(T-t)}\mathcal{N}(d_2).
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k+1) \\
&= \frac{1}{\sqrt{2}} \cdot \frac{x+(k+1)\tau}{\sqrt{\tau}} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\ln(\frac{S}{K}) + (\frac{r}{\sigma^2/2}+1)\frac{\sigma^2}{2}(T-t)}{\sqrt{\frac{\sigma^2}{2}(T-t)}} \\
&= \frac{\ln(\frac{S}{K}) + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}
\end{aligned}
\qquad
\begin{aligned}
d_2 &= \frac{x}{\sqrt{2\tau}} + \sqrt{\frac{\tau}{2}}(k-1) \\
&= \frac{1}{\sqrt{2}} \cdot \frac{x+(k-1)\tau}{\sqrt{\tau}} \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\ln(\frac{S}{K}) + (\frac{r}{\sigma^2/2}-1)\frac{\sigma^2}{2}(T-t)}{\sqrt{\frac{\sigma^2}{2}(T-t)}} \\
&= \frac{\ln(\frac{S}{K}) + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}.
\end{aligned}$$

□

3.3 Convergence of CRR Model to BSM Model

Theorem 3.3.1 (Central Limit Theorem). *Let $X_1, X_2, \cdots$ be a sequence of independent and identically distributed random variables each having mean μ and variance σ^2. Then the distribution of*

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

tends to the standard normal as $n \to \infty$. That is for $-\infty < a < \infty$,

$$\mathbb{P}\left\{\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \leq a\right\} \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-x^2/2}dx$$

as $n \to \infty$.

Theorem 3.3.2 (Berry-Esseen Theorem)**.** *Let* $X_{n,j}$, $n \geq 1$, $1 \leq j \leq k_n$, *be* L^3 *random variables with* $k_n \to \infty$ *such that* $\forall$ n *and* j,

$$\mathbb{E}(X_{n,j}) = 0.$$

Let $F_{n,j}$ *be the cumulative distribution of* $X_{n,j}$.

Write, for $n \geq 1$

$$S_n = \sum_{j=1}^{k_n} X_{n,j}$$

and let $F_n(x)$ *be the cumulative distribution of* S_n.
For $n \geq 1$ *and* $1 \leq j \leq k_n$, *write*

$$\sigma_{n,j}^2 = \mathbb{E}(X_{n,j}^2), \qquad s_n^2 = \sum_{j=1}^{k_n} \sigma_{n,j}^2$$

and

$$\gamma_{n,j} = \mathbb{E}(X_{n,j}^2), \qquad \Gamma_n = \sum_{j=1}^{k_n} \gamma_{n,j}.$$

We further assume that for each n,

$$s_n^2 = \sum_{j=1}^{k_n} \sigma_{n,j}^2 = 1.$$

Then there is some $A_0 < 36$ *such that for each* $n \geq 1$,

$$\sup_{x \in \mathbb{R}} |F_n - \mathcal{N}(x)| \leq A_0 \Gamma_n$$

where $\mathcal{N}(x)$ *is the cumulative normal distribution function.*

Remark 3.3.1. *In the case of a single sequence of independent and identically distributed r.v.'s* $X_j, j > 1$ *with mean* 0, *variance* σ^2, *and third absolute moment* $\gamma < \infty$, *the right side reduces to*

$$\frac{A_0 \gamma}{\sigma^3} \frac{1}{\sqrt{n}}.$$

This special case of the Berry-Eseen Theorem simplifies to

$$\sup_{x \in \mathbb{R}} |F_n - \mathcal{N}(x)| \leq \frac{A_0 \gamma}{\sigma^3} \frac{1}{\sqrt{n}}.$$

In particular where $X_n \sim \mathcal{B}(p)$ and we have that $\forall n$, $\mu = p$ and $\sigma = \sqrt{p(1-p)}$ and

$$
\begin{aligned}
\gamma = E|X_1 - p|^3 &= p(1-p)^3 + (1-p)|0-p|^3 \\
&= p(1-p)^3 + (1-p)p^3 \\
&= p(1-p)[(1-p)^2 - p^2]
\end{aligned} \tag{3.60}
$$

and

$$
\begin{aligned}
\sigma^3 &= (\sqrt{p(1-p)})^3 \\
&= p(1-p)\sqrt{p(1-p)}.
\end{aligned} \tag{3.61}
$$

Definition 9. *We define a binomial random variable*

$$S_n = X_1 + X_2 + \cdots + X_n$$

where $X_i; i \in \{1, 2, \cdots, n\}$ *are Bernoulli random variables* $\mathcal{B}(p)$*.* S_n *is described as the number of successes in* n *Bernoulli trials, where*

$$\mathbb{P}[S_n = k] = \binom{n}{k} p^k (1-p)^{n-k}; \qquad \mathbb{P}[S_n \geq x] = \sum_{k=x}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

and

$$\mathbb{E}[S_n] = \mu = np; \qquad Var[S_n] = \sigma^2 = np(1-p).$$

Definition 10. *Let*

$$S_n^* = \frac{X_1 + X_2 + \cdots + X_n - np}{\sqrt{np(1-p)}}.$$

We define

$$F_n(x) = \mathbb{P}[S_n^* \leq x]$$

alternatively

$$F_n(x) = \mathbb{P}[S_n \leq np + x\sqrt{np(1-p)}].$$

Remark 3.3.2. *We let* $\mathbb{P}[S_n \leq x] = \Psi(x; n, p)$

$$\begin{aligned}\Rightarrow \Psi(x; n, p) &= 1 - \mathbb{P}[S_n \geq x] \\ \Rightarrow \mathbb{P}[S_n \geq x] &= 1 - \Psi(x; n, p) \\ &= 1 - \mathbb{P}\left[S_n^* \leq \frac{x - np}{\sqrt{np(1-p)}}\right].\end{aligned}$$

We also denote $\mathbb{P}[S_n \geq x] = \Phi(x, n, p)$*, hence*

$$\Phi(x; n, p) = 1 - \Psi(x; n, p)$$

Theorem 3.3.3. *Let $C(n, j)$ be the price of a European Call Option at time $t = n\delta t$ on an underlying S with strike price K. Let $C(S, t)$ be the value of the same call option given by the Black-Scholes-Merton formula. We have that*

$$C(S, t) = \lim_{N \to \infty} C(n, j).$$

Proof. This proof closely follows the one given by [48]. We fist recall that from the CRR pricing Formula we have that the value at the nth time step of a European call option with Expiration T and strike K is given by

$$C(n, j) = S(n, j) \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \hat{\pi}^l (1-\hat{\pi})^{N-n-l} - \frac{K}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^l (1-\pi)^{N-n-l}$$

where

$$\hat{k} = \inf\{k \in \mathbb{N} : k > \log(K/(S(t)D^{N-n}))/\log(U/D)\}$$

and

$$\hat{\pi} = \frac{\pi U}{R} \in (0, 1).$$

The call option formula can be rewritten in the following form using the notation previously introduced

$$C(n, j) = S(n, j)\Phi(\hat{k}; M, \hat{\pi}) + KR^{-M}\Phi(\hat{k}; M, \pi)$$

where $M = N - n$.

By the Central Limit Theorem, we have $\Psi(x;n,p) \to \mathcal{N}(x)$ as n $\to \infty$ and thus

$$\begin{aligned}|\Psi(x;n,p) - \mathcal{N}(x)| &= \left|\Psi(x,n,p) - \mathcal{N}\left(\frac{x-np}{\sqrt{np(1-p)}}\right)\right| \\ &= \left|\mathbb{P}[S_n \leq x] - \mathcal{N}\left(\frac{x-np}{\sqrt{np(1-p)}}\right)\right| \\ &= \left|\mathbb{P}\left[S_n^* \leq \frac{x-np}{\sqrt{np(1-p)}}\right] - \mathcal{N}\left(\frac{x-np}{\sqrt{np(1-p)}}\right)\right|\end{aligned}$$

since $\Phi(x;n,p) = 1 - \Psi(x;n,p)$. This implies that

$$1 - \Psi(x;n,p) = \Phi(x;n,p) \to 1 - \mathcal{N}(x) = \mathcal{N}(-x).$$

By the Berry-Eseen Theorem

$$\left|\Phi(\hat{k}, M, p) - \mathcal{N}\left(\frac{Mp - \hat{k}}{\sqrt{Mp(1-p)}}\right)\right| \leq C\frac{p^2 + (1-p)^2}{\sqrt{p(1-p)}}\frac{1}{\sqrt{M}}$$

for $\Phi(\hat{k}, M, \pi)$, where $p = \pi$, we have

$$p = \pi = \frac{e^{r\delta\tau} - e^{-\sigma\sqrt{\delta\tau}}}{e^{\sigma\sqrt{\delta\tau}} - e^{-\sigma\sqrt{\delta\tau}}} \tag{3.62}$$

where $\tau = T - t$ and $\delta\tau = \frac{T-t}{M}$.

Applying the Taylor theorem to 3.62, gives

$$\pi = \frac{r - \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta\tau} + \frac{1}{2} + o(\sqrt{\delta\tau}).$$

From the CRR formula

$$\hat{k} \approx \frac{\ln(\frac{K}{S}) - M\ln(D)}{\ln(\frac{U}{D})} = \frac{\ln(\frac{K}{S}) - M\ln(e^{-\sigma\sqrt{\delta\tau}})}{\ln\left(\frac{e^{\sigma\sqrt{\delta\tau}}}{e^{-\sigma\sqrt{\delta\tau}}}\right)}$$

$$= \frac{\ln(\frac{K}{S}) + M\sigma\sqrt{\delta\tau}}{2\sigma\sqrt{\delta\tau}}$$

$$= \frac{\ln(\frac{K}{S})\sqrt{M}}{2\sqrt{\tau}} + \frac{M}{2}.$$

Therefore

$$\frac{M\pi - \hat{k}}{\sqrt{M\pi(1-\pi)}} \approx \frac{M\left[\frac{r-\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta\tau} + \frac{1}{2}\right] - \left[\frac{\ln(\frac{K}{S})\sqrt{M}}{2\sqrt{\tau}} + \frac{M}{2}\right]}{\sqrt{M}\sqrt{\frac{1}{2}\cdot\frac{1}{2}}}$$

$$= \frac{\left[\frac{M}{2} + \frac{r-\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\tau}\sqrt{M}\right] - \left[\frac{\ln(\frac{K}{S})\sqrt{M}}{2\sqrt{\tau}} + \frac{M}{2}\right]}{\sqrt{M}\sqrt{\frac{1}{4}}}$$

$$= \frac{\frac{r-\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\tau} - \frac{\ln(\frac{K}{S})}{2\sqrt{\tau}}}{\frac{1}{2}}$$

$$= \frac{\ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

$$= d_2.$$

This means $\Phi(\hat{k}; M, \pi) \to \mathcal{N}(d_2)$ as n $\to \infty$. Using the same argument for $\Phi(\hat{k}; M, \hat{\pi})$, where $p =$
$hat\pi$

$$p = \hat{\pi} = \frac{\pi U}{R} = \frac{1 - e^{-(\sigma\sqrt{\delta\tau} + r\delta\tau)}}{1 - e^{-2\sigma\sqrt{\delta\tau}}}.$$

Again by the Taylor Theorem we have:

$$\hat{\pi} = \frac{r + \frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta\tau} + \frac{1}{2} + o(\sqrt{\delta\tau})$$

hence

$$
\begin{aligned}
\frac{M\hat{\pi}-\hat{k}}{\sqrt{M\hat{\pi}(1-\hat{\pi})}} &\approx \frac{M\left[\frac{r+\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\delta\tau}+\frac{1}{2}\right]-\left[\frac{\ln(\frac{K}{S})\sqrt{M}}{2\sqrt{\tau}}+\frac{M}{2}\right]}{\sqrt{M}\sqrt{\frac{1}{2}.\frac{1}{2}}} \\
&= \frac{\left[\frac{M}{2}+\frac{r+\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\tau}\sqrt{M}\right]-\left[\frac{\ln(\frac{K}{S})\sqrt{M}}{2\sqrt{\tau}}+\frac{M}{2}\right]}{\sqrt{M}\sqrt{\frac{1}{4}}} \\
&= \frac{\frac{r+\frac{1}{2}\sigma^2}{2\sigma}\sqrt{\tau}-\frac{\ln(\frac{K}{S})}{2\sqrt{\tau}}}{\frac{1}{2}} \\
&= \frac{\ln(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \\
&= d_1.
\end{aligned}
$$

Which means $\Phi(\hat{k};M,\hat{\pi}) \to \mathcal{N}(d_1)$ as n $\to \infty$ and hence

$$
S\Phi(\hat{k};M,\hat{\pi})+KR^{-M}\Phi(\hat{k};M,\pi) \to S\mathcal{N}(d_1)+Ke^{-r\tau}\mathcal{N}(d_2) \text{ as n} \to \infty
$$

which is the Black Scholes Formula. □

Theorem 3.3.4 (Put-Call Parity)**.** *Given the price of a European call option $C(S,t)$ with strike K and Expiration T, we can determine the price of a put option $P(S,t)$ with the same specifications using the following formula which is called the Put-Call Parity formula,*

$$
C(S,t)-P(S,t)-S(t)+Ke^{-r(T-t)}=0.
$$

3.4 Adjusting For Dividends

In this section we relax the no dividends assumption and allow for intermediate cash flows and show how this is factored into the CRR Binomial Model.

Definition 11. *A dividend is the distribution of reward from a portion of company's earnings and is paid to a class of its shareholders.*

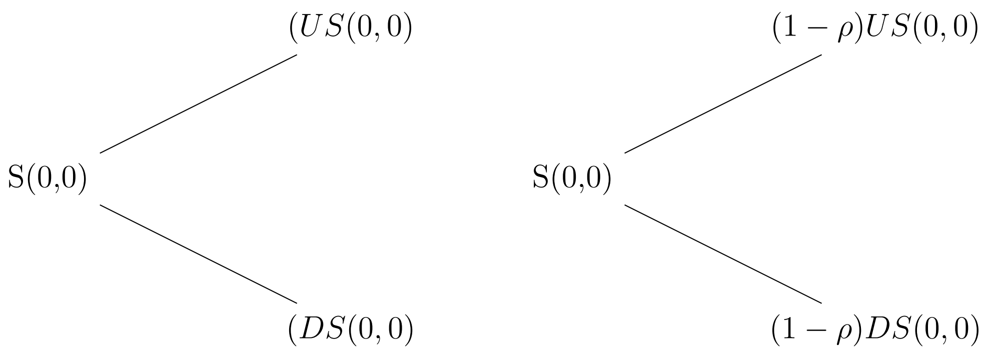
In discussing Dividends, three dates are of particular importance, t_e, t_c, t_p, where $t_e < t_c < t_p$. We describe these dates below:

- t_e is called the **ex-dividend date**, this means if you own a company's stock just before t_e, you are entitled to the dividend cash flow and if you sell the stock just after t_e you are still entitled to the dividend cash flow [48].

- t_c is the date when the company close the books and decide who is entitled to the dividend cash flow [48].
- t_p is when the company sends out the dividend cheques, i.e., pays the dividends [48].

After the **ex-dividend date**, share prices fall by an amount equal to the dividend amount, if this does not occur an arbitrage is created, i.e. you can buy the stock just before t_e and sell it just after t_e for the same amount leading to a riskless profit (dividend amount claimed).

Different companies have different dividend policies, which specifies dividend amount and frequency of dividend payments. We will consider a constant dividend yield policy. We denote $\hat{\delta} = \frac{\hat{D}}{S}$, where $\hat{D}$ is the dividend amount; S is the stock price; and $\hat{\delta}$ is the dividend yield.
We first consider a one period Binomial model and allow for dividends within that single period. The value of the stock at the end of the period take one of two values $(1-\hat{\delta})US$ in the upstate, $(1-\hat{\delta})DS$ in the down state.



(a) Binomial Model without Dividends (b) Binomial Model with Dividends

Figure 3.3: 1-Period Binomial Model with and without Dividends

The dividend does not affect our model in any way except that the stock prices are lower after a dividend payment. In fact the value of a European option with dividends is priced identically as one without dividends, the only difference is that $S(n,j)$ is replaced by $\tilde{S}(n,j) = S(n,j)(1-\hat{\delta})^{\gamma}$, where γ is the number of ex dividend dates. The same holds true in the multi-period case, and we thus have the following corollary.

Corollary 3.4.1. *The arbitrage price of a European call option, $C(N,l) = [\tilde{S}(N,l) - K]^{+}$, written on a stock S with constant dividend yield at the nth time step with expiry time T and strike K in the Binomial Model is given by the CRR option pricing formula*

$$C(n,j) = \tilde{S}(n,j) \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \hat{\pi}^{l}(1-\hat{\pi})^{N-n-l} - \frac{K}{R^{N-n}} \sum_{l=\hat{k}}^{N-n} \binom{N-n}{l} \pi^{l}(1-\pi)^{N-n-l}$$

where

$$\hat{k} = \inf\{k \in \mathbb{N} : k > \log(K/(\tilde{S}(n,j)D^{N-n}))/\log(U/D)\};$$

Input	Impact on Option value
Volatility	The higher the volatility the higher the option value.
Dividend rate	The higher the dividend rate the lower the call option value but the value is increased in the case of a put option.
Interest rate	In the case of a call option value is increased the higher the interest rate whereas the value is diminished in the case of a put option.
Time to expiration	The longer the time to expiration the greater the option value.
Stock price (Underlying asset)	In the case of a call option an increase in the stock price increases value whereas it diminishes value in the case of a put option.
Strike price	The higher the strike price the lower the call option value but the higher the put value.

Table 3.1: The Impact of Option Inputs on Option Value

$$\hat{\pi} = \frac{\pi U}{R} \in (0,1) \qquad and \qquad \tilde{S}(n,j) = S(n,j)(1-\hat{\delta})^{\gamma}.$$

These idea can be extend to continuous time very easily by applying the same methodology to price a European option without dividends but we will not discuss that in this study. We will simply state the price of a European option on an underlying S at time t with expiration date T, strike K and continuous dividend yield $\hat{\delta}$ below:

$$C(S,t) = Se^{-\hat{\delta}t}\mathcal{N}(\hat{d}_1) + Ke^{-r\tau}\mathcal{N}(\hat{d}_2)$$

where:

$$\hat{d}_1 = \frac{\ln\frac{S}{K} + (r - \delta \hat{+} \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \qquad ;\hat{d}_2 \qquad = \frac{\ln\frac{S}{K} + (r - \delta \hat{-} \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

The inputs of a model impact the values of its output, it is important to know how input variables affect the output values, we give the impact of option inputs on option value in table 3.1 below.

3.5 American Options

In this section we present the valuation of the American style call option in discrete time. Since an American call option can be exercised at any time upto and including the maturity date, at each time step an American option can be sold, held or exercised. This means if $V(n,j)$

denotes the value of an American option then,

$$V(n,j) \geq [S(n,j) - K]^+.$$

$V(n,j)$ can be computed by a simple adjustment to the backwardization formula. At node (n,j) we first calculate

$$W(n,j) = \frac{\pi V(n+1,j+1) + (1-\pi)V(n+1,j)}{R}$$

and compare $W(n,j)$ with $[S(n,j)-K]^+$. If $W(n,j) > [S(n,j)-K]^+$ then we hold the option. However, if $W(n,j) < [S(n,j) - K]^+$ then we exercise the option. Hence,

$$V(n,j) = \max(W(n,j), [S(n,j) - K]^+).$$

The algorithm then becomes,

$$\begin{aligned} V(N,j) &= [S(n,j) - K]^+ \\ W(n,j) &= \frac{\pi V(n+1,j+1) + (1-\pi)V(n+1,j)}{R} \\ V(n,j) &= \max(W(n,j), [S(n,j) - K]^+) \\ C(0,0) &= V(0,0). \end{aligned}$$

We thus have that the value of an American style option $\geq$ the value of a European style option.

Chapter 4

Real Options

The failure of DCF methods as discussed in the preceding sections and chapters is its inability to incorporate the impact of managerial flexibility on project value. The root cause of this inadequacy can be found clearly in the NPV formula which suggests that real projects are analogous to a portfolio of riskless bonds. [33] asserted that the treatment of an investment project as a portfolio of riskless bonds in the presence of uncertainty and managerial flexibility was inadequate and highly suspect. Consequently [27] suggested the use of contingent claim analysis in order to capture the extra component of value. He stated that it would be more accurate to view investment projects as being analogous to financial options since they both have asymmetric returns. The analogy is intuitive, [11] described investment opportunities as options, i.e., rights to take a certain action (buy or sell an asset) without the symmetric obligation to perform the action at a future date [47]. The payoffs of financial options are asymmetric as are the returns of capital investment projects with operating flexibility. Thus [33] likened the use of DCF analysis to valuing a stock option and ignoring the asymmetric right to take or not to take an action. They further stated

> "while traditional analysis views capital investment, like children, as hostages to fortune, our approach (*of viewing investment as options*) recognizes that, like children also, they are often amenable to their progenitors' guidance." (*emphasis my own*)

The option analogy has intuitive appeal but there are differences between financial options and "real options" as Stewart C. Meyers called them. We discuss the similarities and differences in table 4.1

	Financial Options	Real Options
Option Price	Price paid to acquire option	Price paid to acquire option, keep it alive, and clear uncertainty. Negotiable
Asset Price	Underlying Asset/share price	PV given by DCF
Strike Price	Agreed	Investment Cost
Expiration Time	Defined, known	Known and Unknown
Timing of payoff	Instantaneous	Often quite after expiry, over a long time.
Volatility	Of underlying Asset	Of main future cash flow driver
Time value of money	Interest rate	WACC, treasury rate, etc
Resolution of uncertainty	Automatic	Not Automatic
Option holder's control on value	None	Mostly

Table 4.1: Finacial Vs Real Options

It is worth noting that this analysis is not a replacement of DCF analysis but rather an augmentation. The purpose of real options analysis is to capture the option value component of the expanded NPV. [45] demonstrates that in the absence of managerial flexibility the options framework yields the same value as the static DCF, DCF can then be considered as a special case of real options analysis (ROA).

ROA solves the discount rate issue inherent inherent in DTA. In ROA we discount using the risk-free rate since risk is embedded in the up and down factors as shown in the previous chapter on options pricing (CRR model) hence no need to determine a new discount rate to reflect the change in risk profile.

The ROA literature is vast, tables 4.2 and 4.3 describe the different types of options that are encountered most frequently, the researchers that analysed them as well as industries they are important in, the table was adopted from [47].

Category	Description	Important In	Analysed By
Option to defer	Management holds a lease on (or option to buy) valuable land resources. It can wait (x years) to see if output prices justify constructing a building or plant, or developing a field.	All natural resource extraction industries; real estate development; farming; paper products.	Tourinho [43]; Titman [42]; McDonald & Seigel [26]; Paddock, Seigel & Smith [16]; Ingersoll and Ross [18].
Time to build option (staged investment)	Staging investment as a series of outlays creates an option to abandon the enterprise midstream if new information is unfavourable. Each stage can be viewed as an option on subsequent stages, and valued as a compound option.	All R&D intensive industries, especially pharmaceuticals; long development capital intensive projects e.g. large scale construction or energy generating plants; start-ventures.	Majd & Pindyck [25]; Carr [4]; Trigeorgis
Option to alter operating scale (e.g., to expand; to contract; to shut down & restart)	If market conditions are more favourable than expected, the form can expand the scale of production or accelerate resource utilization. Conversely if conditions are less favourable than expected, it can reduce the scale of operations. In extreme conditions, production may temporarily halt and start again.	Natural resource industies such as mine operations; facilities planning and construction in cyclical industries; fashion apparel; consumer goods; commercial real estate.	Brennan & Schwartz [1]; McDonald & Seigel [26]; Trigeorgis & Mason [45]; Pindyck [34].
Option to abandon	If market conditions decline severely, management can abandon current operations permanently and realize the resale value of capital equipment and other assets in the secondhand markets.	Capital intensive industries such as airlines and railroads; financial services; new product introductions in uncertain markets	Myers & Majd [28].

Table 4.2: Summary of Common Real Options and Industry Applications 1

Category	Description	Important In	Analysed By
Option to switch (e.g., outputs and inputs)	If prices or demand change, management can change the output mix of the facility (product flexibility). Alternatively, the same outputs can be produced using different types of inputs (process flexibility).	*Output shifts*: any good sought in small batches or subject to volatile demand, e.g., consumer electronics; toys specialty paper; machine parts; autos. *Input shifts*: all feedstock dependent facilities, e.g., oil; electric power; chemicals; crop switching; sourcing.	Magrabe [24]; Kensinger [19]; Kulatilaka [21]; Kulatilaka & and Trigeorgis [22]
Growth options	An early investment (e.g., R&D, lease on undeveloped land or oil acquisition, information network/infrastructure) is a prerequisite or link in a chain of interrelated projects, opening up future growth opportunities (e.g., new generation product or process, oil reserves, access to new market, strengthening of core capabilities). Like inter-project compound options.	All infrastructure based or strategic industries, especially high-tech, R&D or industries with multiple project generations or applications (e.g., computers and pharmaceuticals); multinational operations; strategic acquisitions.	Myers [27]; Brealy & Myers [32]; Kester [20]; Trigeorgis [44]; Pindyck [34]; Chung & Charoenwong [6].
Multiple interacting options	Real life projects often involve a collection of various options, both upward -potential enhancing calls and downward-protection put options present in combination. Their combined option value may differ from he sum of separate option values, i.e., they interact. They may also interact with financial flexibility options.	Real-life projects in most industries discussed above.	Brennan & Schwartz [33]; Trigeorgis [46]

Table 4.3: Summary of Common Real Options and Industry Applications 2

4.1 Option to Invest

As we previously noted, most investment projects are not now or never propositions as suggested by the DCF methodology, but management has the ability to defer investment. Since projects can be deferred this means that an investment project competes with itself delayed in time [18]. Managements may choose to defer investment because:

- Current market conditions may be unfavourable and current NPV < 0;
- NPV > 0 but there may be too much uncertainty in the market and therefore prudent to wait and see how things turn out [11].

The second reason for waiting stems from the fact that investment expenditures are at the very least partially irreversible which means that if the market turns south, then these expenditures are sunk costs [34]. Since investment involves some degree of irreversibility, it is valuable to wait for the arrival of new information, i.e., the resolution of uncertainty. When a firm chooses to invest they are effectively giving up the possibility of waiting for new information which might be pivotal to the desirability and timing of the investment [34]. This means that investing involves an opportunity cost which is not factored into the NPV calculation. Hence, "the correct calculation involves comparing the value of investing today with the (present) value of investing at all possible times in the future [26]". An investment project is thus better treated as a call option on the PV of the project with exercise price, I, the investment outlay [44]. Uncertainty, irreversibility and managerial flexibility interact to give deferrability value (option to invest) [10]. Traditional NPV analysis assumes that if projects are not taken immediately they disappear. NPV is therefore myopic in its view of capital investment since the opportunity cost of investment is ignored. The NPV rule therefore needs to be altered, instead of, invest if NPV > 0, management should invest if NPV $>$ option to invest or if PV $>$ investment outlay and opportunity cost of investing. Although waiting has its merits, the investor gives the cash-flows that could have been otherwise earned during the waiting period, this is treated as a dividend also called (leakage) in ROA. Competition is another factor that should be considered when analyzing the value of waiting, as competitive entry will diminish the profits and market share of the incumbent firm [47]. Consequently the benefits of waiting must be weighed against the costs and the optimal decision made accordingly [35]. In light of this [44] suggested a real options classification scheme based on the following strategic questions:

1. Is the option exclusive and what is the impact of competition on the incumbent firm's ability to appropriate option value?
2. Is the investment opportunity valuable in and of itself or is it a link in a chain of subsequent investment opportunities?
3. Does the project require an immediate accept/reject decision or can the decision be deferred?

The first strategic question deals with the exclusivity of the rights of a project, if rights are exclusive it is called proprietary, if not, we refer to them as shared. The second strategic question is concerned with the idea presented by Myers which he referred to as the times series inter-dependencies, it categorises the option into simple and compound, it is called a compound option if the investment is a link to subsequent investment and simple if not. The third question deals with whether projects have an expiration date attached to them or not, it categorises projects into deferrable and expiring. We present the option classification scheme in figure 4.1 below.

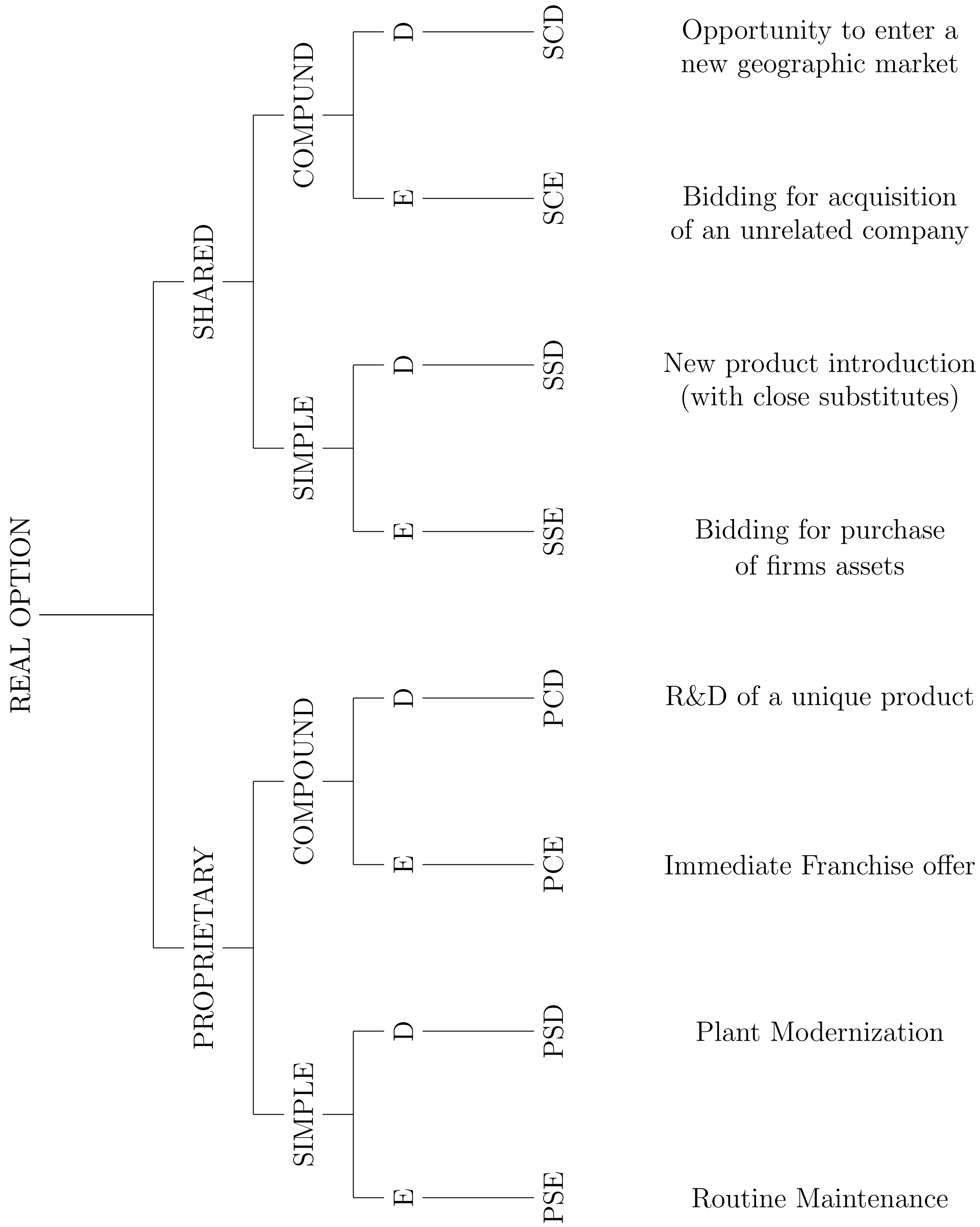


Figure 4.1: Real Options Classification Scheme

The subject of shared real options and the impact of competition on option value can be found in [47], [40], [39] and [5]. Another factor to consider is the impact of interest rate uncertainty on project value, this is discussed by [18] and [38]. For our purposes however, we will operate under the CRR assumption of constant interest rates. In the next chapter, we analyse an option to invest in a basalt quarry by Mbebane enterprises, the option is proprietary since they hold a lease to the quarry. This is a PSD, according to the classification scheme.

Chapter 5

Data Analysis and Results

In this chapter we analyse an investment opportunity by Mbebane Enterprises of a Basalt quarry for which they holds the rights. The project will take $1 - year$.

Methodology

We summarise the analysis below:

Step 1 We first carry out a DCF to determine the static NPV portion of the expanded NPV.

Step 2 We will then use logarithmic cash flows method to determine the volatility of the projects cash flows.

Step 3 With the volatility determined, we can determine up and down factors and the risk neutral probabilities of CRR model.

Step 4 Use the CRR model to determine option value.

5.1 Discounted Cash-flow (Mbebane Enterprises Basalt Quarry)

The DCF analysis was carried out by Mbebane Enterprises. We summarise the DCF analysis in the table below:

Month	Cash-flow	Discount Factor	Present Value
0	BWP 1,305,000.00	1	BWP 1,305,000.00
1	BWP 385,351.00	0.995850622	BWP 383,752.03
2	BWP 987,901.00	0.991718462	BWP 979,719.66
3	BWP 973,861.00	0.987603448	BWP 961,788.48
4	BWP 501,181.00	0.983505508	BWP 492,914.27
5	BWP 973,861.00	0.979424572	BWP 953,823.39
6	BWP 987,901.00	0.97536057	BWP 963,559.68
7	BWP 487,141.00	0.971313431	BWP 473,166.60
8	BWP 987,901.00	0.967283085	BWP 955,579.93
9	BWP 973,861.00	0.963269462	BWP 938,090.56
10	BWP 501,181.00	0.959272493	BWP 480,769.15
11	BWP 973,861.00	0.955292109	BWP 930,321.73
12	BWP 987,901.00	0.951328242	BWP 939,818.12
RADR		0.05	
PV OF CASHFLOWS			**BWP 9,453,303.61**
NPV			**BWP 8,148,303.61**
IRR			54%

Table 5.1: DCF Summary of Mbebane Basalt Quarry

Since NPV > 0, according to the NPV rule we should invest. The IRR > RADR, thus by IRR rule we should invest. Both DCF metrics agree. We then carry out ROA to determine whether the immediate investment is optimal or waiting is appropriate. To determine the option value, we first need to determine the volatility of cash-flows using logarithmic cash-flow method. The calculations were performed in excel, the table is given below.

Volatility Estimation

Month	Cash-Flow	$\ln(R_t) = S_t/S_{t-1}$	$\ln R_t$
0	BWP 1,305,000.00		
1	BWP 385,351.00	0.295288123	-1.219803712
2	BWP 987,901.00	2.563639383	0.941427883
3	BWP 973,861.00	0.98578805	-0.014313907
4	BWP 501,181.00	0.514632992	-0.66430127
5	BWP 973,861.00	1.943132321	0.66430127
6	BWP 987,901.00	1.014416842	0.014313907
7	BWP 487,141.00	0.493107103	-0.707028881
8	BWP 987,901.00	2.027956998	0.707028881
9	BWP 973,861.00	0.98578805	-0.014313907
10	BWP 501,181.00	0.514632992	-0.66430127
11	BWP 973,861.00	1.943132321	0.66430127
12	BWP 987,901.00	1.014416842	0.014313907
Average $\ln R_t$			-0.023197986
Periodic Volatility σ_p			68%
Annualized Volatility σ			237%

Table 5.2: Logarithmic Cash-flow

The cash-flows of this project are highly volatile with a volatility of 237%.

5.2 Option to Invest (Basalt Quarry)

In this section we use CRR model to determine option value and timing of investment. We analyse this investment opportunity as an American call option. The present value of cash-flows is treated as the value of the underlying, the initial investment outlay is treated as the strike price, we assume a zero leakage (dividend) rate. Table 5.3 lists the values of input and option parameters. The annualized volatility calculated using the logarithmic cash-flow method is used to determine the up and down factors, which are then used to calculate the risk neutral probabilities.

Table 5.3: Parameters

(a) Input Parameters

Parameter	Value
S	BWP 8,148,303.61
K	BWP 1,305,000.00
r	0.0475
σ	237%
T	1
dt	0.125
leakage rate	0

(b) Option Parameters

Parameter	Value
U	2.308680981
D	0.433147762
π	0.305410426
$1 - \pi$	0.694589574
R	1.005955162
1/R	0.994080092

Using the the present value of cash-flows and the up and down factors, the asset lattice (blue nodes) is generated. We then use the American call option backwardization algorithm to generate the option lattice (green nodes) and option value is determined. The orange nodes are the binomial probabilities associated with each outcome across time. The lattices are shown below.

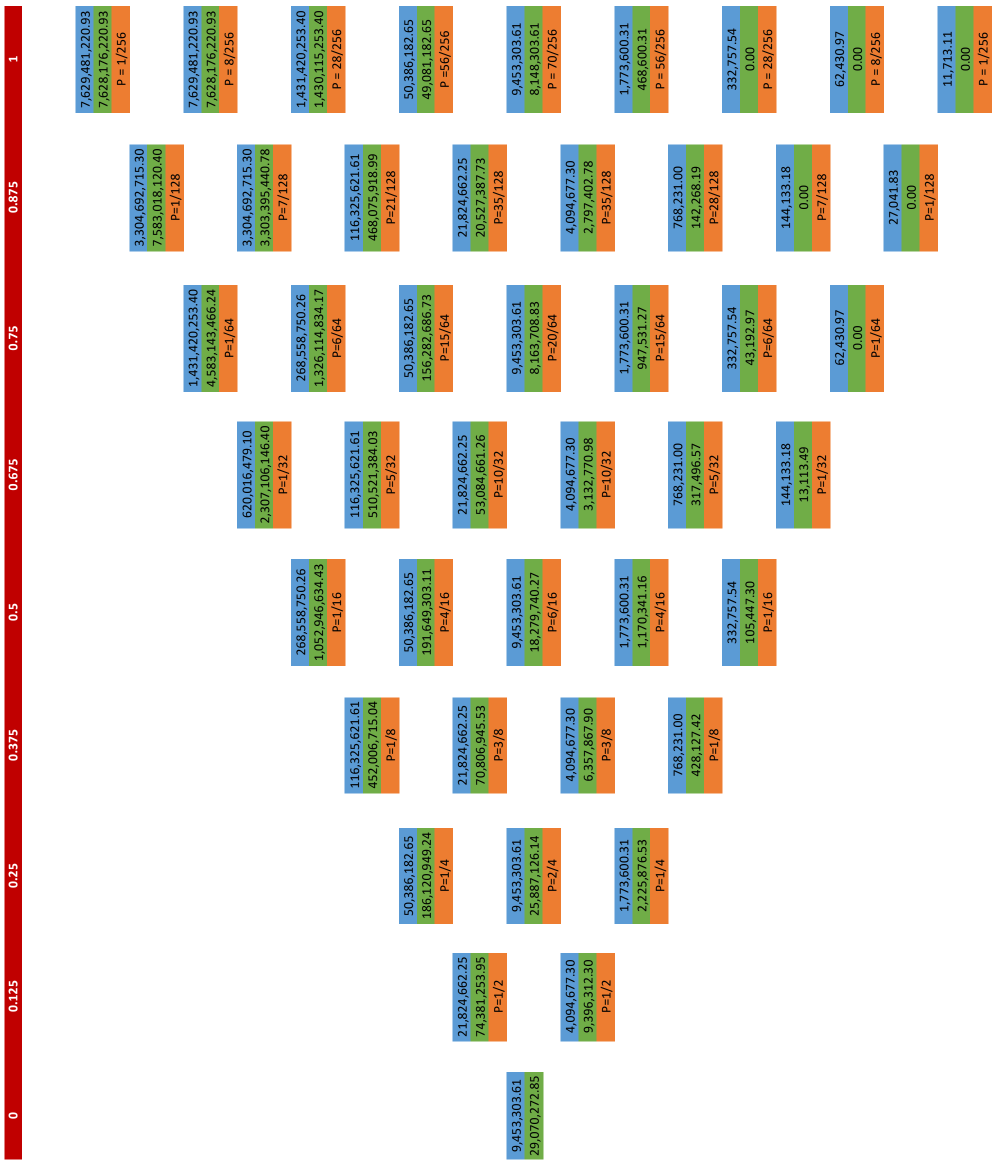


Figure 5.1: CRR Binomial Lattice

The expanded NPV obtained from CRR Binomial lattice is BWP 29,070,272.85. This means that the option value is BWP 20,921,969.24. Since option value is > NPV, according to option analysis it is not optimal to exercise the option (i.e. wait), but NPV analysis and IRR metric make this a go project. The diagrams below indicate regions where exercising the option would result in net gain, i.e., (+)NPV and where exercising the option would result in net loss, i.e., (-)NPV.

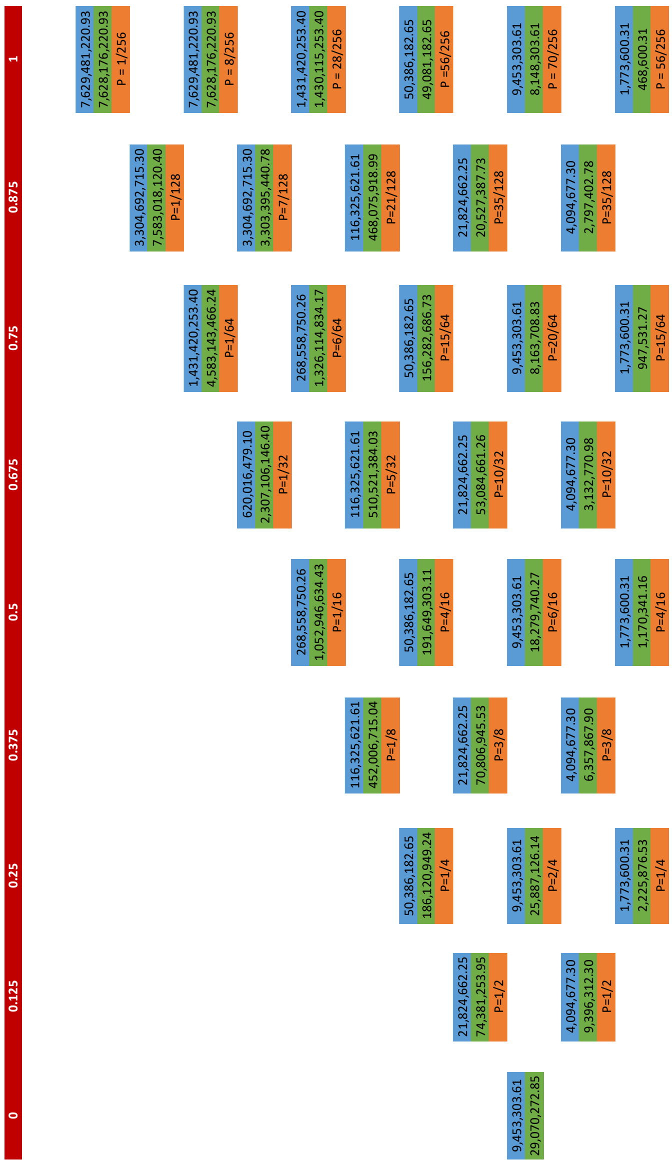


Figure 5.2: (+)NPV Region

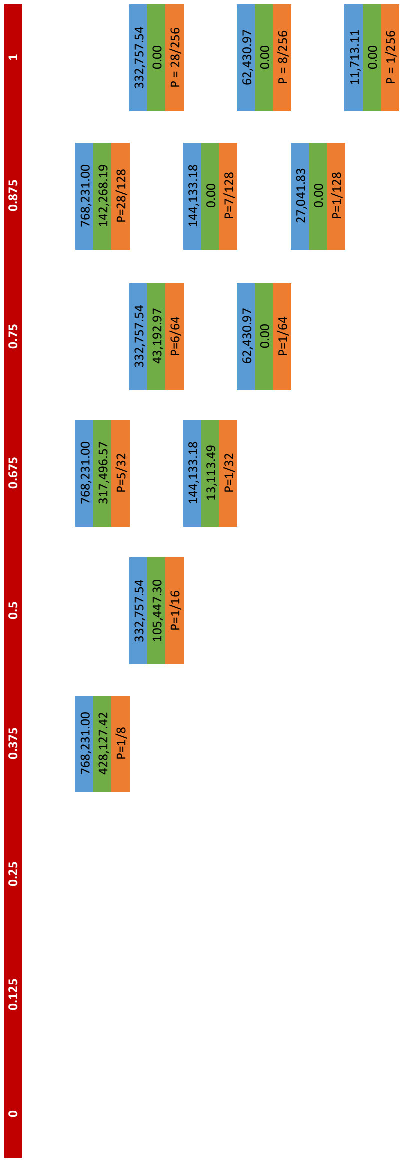


Figure 5.3: (-)NPV Region

The (-)NPV region is smaller than the (+)NPV region, hence management may want to wait so they can realise greater value, but it does come at a risk. So the decision to wait will be dependent on the risk appetite of management. In the (+)NPV region option value > NPV hence it is never optimal to exercise early but option value is unrealised value, hence management will have to consider exercising or waiting based on probabilities of gains or losses in value. The probabilities, as well as the max and min values of cash-flows at each point in time are shown in the binomial lattices constructed. Swings in the the NPV are very high due to the high volatility value, this should also be considered when making the decision to wait or invest.

Time	Probability of (+) Returns
0	100%
0.125	100%
0.25	100%
0.375	88%
0.5	94%
0.675	78%
0.75	89%
0.875	77%
1	86%

Table 5.4: Probability of (+) Returns

The decision to invest is thus left to the risk preference of management which is guided by the probability assigned by the binomial distribution at each node. Although the decision to invest early is not optimal it might be more profitable than the optimal decision to exercise the option at expiry. Hence the lattice can be used as a decision tool to make this all important decision.

Chapter 6

Conclusion and Recommendations

DCF methods as we discussed in this dissertation are not adequate in capturing all sources of value inherent in many projects especially when project value is highly volatile, investment costs are irreversible and management has the capacity to make mid-course corrections. The presence of these 3 factors introduces option value. Option value is not captured by standard capital measurement techniques but is captured well by contingent claim analysis. The option to invest (wait) is a source of value inherent in many projects and should always be considered when appraising and timing investment projects. In this dissertation we analysed an investment project by Mbebane Enterprises for a basalt quarry, DCF techniques made this a 'go' project. ROA was then used to determine whether immediate investment was appropriate, we found that option value exceeded NPV hence immediate investment was found not to be optimal, the optimal decision was to wait until expiration but this could mean loss in project value. The binomial lattice presented graphically the max and min values of cash-flows and attached probabilities to them which can be used as guides for making the investment decision. Our analysis showed that option value was very significant this was due to the high volatility of cash-flows, this meant large swings in project value, hence waiting could bring significant gains in value or significant loss in value. We found that the binomial lattice could thus be used to graphically show these swing and probability of returns. Using the lattice, management could make a more informed decision on waiting or immediate investment based on their risk preferences.

In conclusion, ROA is a powerful tool that arms managers with a better appreciation of value inherent in projects. We therefore recommend that ROA should be used in conjunction with standard methods especially where volatility and managerial flexibility are significant so as to avoid undervaluing projects and thus leaving potentially highly profitable projects. It is also very useful for timing investment as was demonstrated in our analysis of the basalt quarry. The Real Options methodology equips managers with a stochastic perspective on project value, which translates into better valuation. ROA is a powerful tool but not omniscient and should thus be used circumspectly.

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